# ANALYSIS 1
## Course & Exercises

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Mai 2025

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# Contents

# Chapter 1

# Real numbers and complex numbers

## 1.1 The set of real numbers

### 1.1.1 Commutative field structure

Let $\mathbb{R}$ be the set of real numbers.

We provide $\mathbb{R}$ with two operations: addition $(+)$ and multiplication $(.)$.

$(\mathbb{R}, +, .)$ is a commutative field:

1) $(+)$ is internal: $\forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$.

2) $(+)$ is commutative: $\forall x, y \in \mathbb{R}, x + y = y + x$.

3) $(+)$ is associative: $\forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$.

4) $(+)$ admits a neutral element: $\exists e = 0 \in \mathbb{R}, \forall x \in \mathbb{R}, x + 0 = 0 + x = x$.

5) Every element has a symmetrical element:
$\forall x \in \mathbb{R}, \exists x' = -x \in \mathbb{R}, x + x' = x' + x = 0$.

So, $(\mathbb{R}, +)$ is a commutative group.

6) $(.)$ is internal: $\forall x, y \in \mathbb{R}, x.y \in \mathbb{R}$.

7) $(.)$ is commutative: $\forall x, y \in \mathbb{R}, x.y = y.x$.

8) $(.)$ is associative: $\forall x, y, z \in \mathbb{R}, (x.y).z = x.(y.z)$.

9) Distributivity: $\forall x, y, z \in \mathbb{R}, (x + y).z = x.z + y.z$.

So, $(\mathbb{R}, +, .)$ is a commutative ring.

10) $(.)$ admits a neutral element: $\exists e' = 1 \in \mathbb{R}, \forall x \in \mathbb{R}, x.1 = 1.x = x$.

11) Every non-zero element has a symmetric element:
$\forall x \in \mathbb{R}^*, \exists x' = x^{-1} \in \mathbb{R}, x.x^{-1} = x^{-1}.x = 1$.

Hence, $(\mathbb{R}+, .)$ is a commutative field.

### 1.1.2 Total order relation

We provide $\mathbb{R}$ with a total order relation $(\leq)$:

*a*) $(\leq)$ is reflexive: $\forall x \in \mathbb{R}, x \leq x$.
*b*) $(\leq)$ is antisymmetric: $\forall x, y \in \mathbb{R}, (x \leq y \wedge y \leq x) \Rightarrow x = y$.
*c*) $(\leq)$ is transitive: $\forall x, y, z \in \mathbb{R}, (x \leq y \wedge y \leq z) \Rightarrow x \leq z$.
*d*) $(\leq)$ is a total order relation: $\forall x, y \in \mathbb{R}, x \leq y \vee y \leq x$.
i.e. we can always compare between two real numbers.
*e*) $\forall x, y, z \in \mathbb{R}, x \leq y \Rightarrow x + z \leq y + z$.
*f*) $(0 \leq x \wedge 0 \leq y) \Rightarrow 0 \leq x.y$.

**Conclusion 1** $(\mathbb{R}, +, \times, \leq)$ *is a totally ordered commutative field.*

### 1.1.3 The absolute value

**Notation 2** $\mathbb{R}^* = \mathbb{R} \backslash \{0\}$.
$\mathbb{R}_+ = \{x \in \mathbb{R} / x \geq 0\}$, $\mathbb{R}_+^* = \{x \in \mathbb{R} / x > 0\}$.
$\mathbb{R}_- = \{x \in \mathbb{R} / x \leq 0\}$, $\mathbb{R}_-^* = \{x \in \mathbb{R} / x < 0\}$.
$\mathbb{R}_+ \cup \mathbb{R}_- = \mathbb{R}$, $\mathbb{R}_+ \cap \mathbb{R}_- = \{0\}$.

**Definition 3** *The absolute value is the following application:*

$$\begin{array}{rccl} |.|: & \mathbb{R} & \longrightarrow & \mathbb{R}_+ \\ & x & \longrightarrow & |x| = \left\{ \begin{array}{ll} x, & x \geq 0, \\ -x, & x < 0. \end{array} \right. \end{array}$$

**Remark 4** $\forall x \in \mathbb{R}, \ -|x| \leq x \leq |x|$.

**Absolute value properties:**
1) $|x| = 0 \Longleftrightarrow x = 0$.
2) $\forall x, y \in \mathbb{R}, \ |x.y| = |x|\,|y|$.
3) $\forall x \in \mathbb{R}^*, \left|\frac{1}{x}\right| = \frac{1}{|x|}$.
4) $\forall x \in \mathbb{R}, \ (|x| \leq \alpha \Longleftrightarrow -\alpha \leq x \leq \alpha)$.
5) $\forall x, y \in \mathbb{R}, \ |x \pm y| \leq |x| + |y|$ :Triangle inequality.
6) $\forall x, y \in \mathbb{R}, \ ||x| - |y|| \leq |x \pm y|$.

### 1.1.4 Intervals of $\mathbb{R}$

**Definition 5** *Let $I$ be a nonempty part of $\mathbb{R}$. $I$ is an interval of $\mathbb{R}$ if it satisfies the following property:*

$$\forall x_1, x_2 \in I, \forall y \in \mathbb{R}, (x_1 < y < x_2 \Longrightarrow y \in I).$$

**Proposition 6** *Let $I_1$ and $I_2$ be two intervals of $\mathbb{R}$ such that $I_1 \cap I_2 \neq \emptyset$, then $I_1 \cap I_2$ is an interval of $\mathbb{R}$.*

Indeed, let $x_1, x_2 \in I_1 \cap I_2$ be such that $x_1 < x_2$ and $y$ such that $x_1 < y < x_2$.
Let us show that $y \in I_1 \cap I_2$ :
We have $x_1, x_2 \in I_1 \cap I_2 \Longrightarrow x_1, x_2 \in I_1 \wedge x_1, x_2 \in I_2$,
therefore, $x_1, x_2 \in I_1 \wedge x_1 < y < x_2 \Longrightarrow y \in I_1$ (because $I_1$ is an interval),
and $x_1, x_2 \in I_2 \wedge x_1 < y < x_2 \Longrightarrow y \in I_2$ (because $I_2$ is an interval),
hence, $y \in I_1 \wedge y \in I_2 \Longrightarrow y \in I_1 \cap I_2$.
So, $I_1 \cap I_2$ is an interval.

**Remark 7** *The union of two intervals of* $\mathbb{R}$ *is not necessarily an interval.*

Here is an example:

**Example 8** $I_1 = [0,1] \wedge I_2 = [3,4]$.
$I = I_1 \cup I_2 = [0,1] \cup [3,4]$.
*We take* $x_1 = 1 \in I$ , $x_2 = 3 \in I$ *and* $y = 2$.
*We have* $x_1 = 1 < y = 2 < x_2 = 3$, *but* $y = 2 \notin I$.
*Hence,* $I = I_1 \cup I_2$ *is not an interval.*

**The intervals of** $\mathbb{R}$ :
Let $a, b \in \mathbb{R}$ be such that $a \leq b$,

$[a,,b] = \{x \in \mathbb{R}/a \leq x \leq b\}$,

$[a,b[ = \{x \in \mathbb{R}/a \leq x < b\}$,

$]a,b] = \{x \in \mathbb{R}/a < x \leq b\}$,

$]a,b[ = \{x \in \mathbb{R}/a < x < b\}$,

$[a,+\infty[ = \{x \in \mathbb{R}/a \leq x\}$,

$]a,+\infty[ = \{x \in \mathbb{R}/a < x\}$,

$]-\infty,a] = \{x \in \mathbb{R}/x \leq a\}$,

$]-\infty,a[ = \{x \in \mathbb{R}/x < a\}$,

$\mathbb{R} = ]-\infty,+\infty[$.

### 1.1.5 Upper bound of a subset of $\mathbb{R}$

**Definition 9** *Let* $A$ *be a nonempty part of* $\mathbb{R}$ $(A \subset \mathbb{R}, A \neq \emptyset)$. *We say that* $M \in \mathbb{R}$ *is an upper bound of* $A$ *if it satisfies*

$$\forall x \in A,\ x \leq M$$

*We say that* $A$ *is bounded above if*

$$\exists M \in \mathbb{R}, \forall x \in A,\ x \leq M$$

**Remark 10** *If* $M$ *is an upper bound of* $A$, *then all real numbers* $M' \geq M$ *are also upper bounds of* $A$.

### 1.1.6 Supremum of a subset of $\mathbb{R}$

**Definition 11** *Let $A$ be a nonempty part of $\mathbb{R}(A \subset \mathbb{R}, A \neq \emptyset)$ and bounded above. The supremum of $A$ is the smallest upper bound of $A$. We denote it* $\sup A$.

**Remark 12** *The supremum when it exists is unique.*

**Example 13** $A = [-1, 5] \subset \mathbb{R},\ A \neq \emptyset, \forall x \in A,\ x \leq 5,$
*$M = 5$ is an upper bound of $A$.*
*We notice that all the real $M' \geq 5$ are also upper bounds of $A$. Then, the set of upper bounds of $A$ is $M_A = [5, +\infty[$.*
$\sup A$ *is the smallest upper bound of $A$, then* $\sup A = 5$.

### 1.1.7 Lower bound of a subset of $\mathbb{R}$

**Definition 14** *Let $A$ be a nonempty part of $\mathbb{R}$ $(A \subset \mathbb{R}, A \neq \emptyset)$. We say that $m \in \mathbb{R}$ is a lower bound of $A$ if it satisfies*

$$\forall x \in A,\ m \leq x$$

*We say that $A$ is bounded below if*

$$\exists m \in \mathbb{R}, \forall x \in A,\ m \leq x$$

**Remark 15** *If $m$ is a lower bound of $A$, then all real numbers $m' \leq m$ are also lower bounds of $A$.*

### 1.1.8 Infimum of a subset of $\mathbb{R}$

**Definition 16** *Let $A$ be a nonempty part of $\mathbb{R}(A \subset \mathbb{R}, A \neq \emptyset)$ and bounded below. The infimum of $A$ is the greatest lower bound of $A$. We denote it* $\inf A$.

**Remark 17** *The infimum when it exists is unique.*

**Example 18** $A = [-1, 5] \subset \mathbb{R},\ A \neq \emptyset,\ \forall x \in A,\ -1 \leq x,$
*$m = -1$ is a lower bound of $A$.*
*We notice that all the real $m' \leq -1$ are also lower bounds of $A$.Then the set of lower bound of $A$ is $m_A = ]-\infty, -1]$.*
$\inf A$ *is the greatest lower bound of $A$, then* $\inf A = -1$

**Remark 19** 1) *If $A$ is a nonempty part of $\mathbb{R}$ not bounded above, we set* $\sup A = +\infty$.
2) *If $A$ is a nonempty part of $\mathbb{R}$ not bounded below, we set* $\inf A = -\infty$.

### 1.1.9 Bounded set

**Definition 20** *Let $A$ be a nonempty part of $\mathbb{R}$. $A$ is bounded if $A$ is bounded above and bounded below.*

$$A \text{ is bounded } \iff (\exists M, m \in \mathbb{R}, \forall x \in A,\ m \leq x \leq M).$$

$$A \text{ is bounded } \iff (\exists \alpha > 0, \forall x \in A,\ |x| \leq \alpha).$$

### 1.1.10 Axiom of the supremum

Any nonempty part of $\mathbb{R}$ and bounded above admits a supremum.

### 1.1.11 Axiom of the infimum

Any nonempty part of $\mathbb{R}$ and bounded below admits an infimum.

**Remark 21** *This result is not true in $\mathbb{Q}$.*

*Indeed, let $A$ be a nonempty and bounded part of $\mathbb{Q}$, then $\sup A$ and $\inf A$ do not necessarily exist in $\mathbb{Q}$.*

**Example 22** $A = \left[-\sqrt{2}, \sqrt{2}\right] \cap \mathbb{Q}$ *is a nonempty and bounded part of $\mathbb{Q}$. We notice that $\sup A = \sqrt{2} \notin \mathbb{Q}$ and $\inf A = -\sqrt{2} \notin \mathbb{Q}$*

### 1.1.12 Characterization of the supremum

**Proposition 23** *Let $A$ be a nonempty part of $\mathbb{R}$ and let $M \in \mathbb{R}$. Then we have*

$$M = \sup A \iff \begin{cases} 1^\circ)\ \forall x \in A,\ x \leq M, \\ 2^\circ)\ \forall \varepsilon > 0, \exists x_0 \in A \ /\ x_0 > M - \varepsilon. \end{cases}$$

**Example 24** *Let $A = [1, 2[\,,\quad M = 2,\ \forall x \in A, x \leq 2.$*

*If we take $\varepsilon = 0,1 \Longrightarrow M - \varepsilon = 1,9.$*

$\exists x_0 = 1,94 \in A \ /\ x_0 = 1,94 > M - \varepsilon = 1,9.$

*Then $(M - \varepsilon)$ is not an upper bound of $A$.*

### 1.1.13 Characterization of the infimum

**Proposition 25** *Let $A$ be a nonempty part of $\mathbb{R}$ and let $m \in \mathbb{R}$. Then we have*

$$m = \inf A \iff \begin{cases} 1^\circ)\ \forall x \in A,\ x \geq m, \\ 2^\circ)\ \forall \varepsilon > 0, \exists x_0 \in A \ /\ x_0 < m + \varepsilon. \end{cases}$$

**Remark 26** *If $M$ is an upper bound of $A$ and $M \in A$, then $M = \sup A$.*
*Indeed, let $\varepsilon > 0$, just take $x_0 = M > M - \varepsilon$,*
*from the characterization of the supremun, $M = \sup A$.*
*In this case $\max A = \sup A = M$ ($\max A$ exists).*

**Remark 27** *If $m$ is a lower bound of $A$ and $m \in A$, then $m = \inf A$.*
*Indeed, let $\varepsilon > 0$, just take $x_0 = m < m + \varepsilon$,*
*from the characterization of the infimum, $m = \inf A$.*
*In this case $\min A = \inf A = m$ ($\min A$ exists).*

**Remark 28** *- If $\sup A$ exists but $\sup A \notin A$, then $\max A$ do not exists.*
*- If $\inf A$ exists but $\inf A \notin A$, then $\min A$ do not exists.*

**Example 29** *Let $A = [0, 1[$,*
*$\sup A = 1 \notin A$, then $\max A$ do not exists,*
*$\inf A = 0 \in A$, then $\min A = \inf A = 0$.*

### 1.1.14 Properties of the supremum and the infimum

Let $A$ and $B$ de two nonempty parts of $\mathbb{R}$. Then we have the following properties :

1) If $B$ is bounded and $A \subset B$, then $A$ is bounded and we have

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

If $A$ and $B$ are bounded, then

2) $A \cup B$ is bounded and nonempty and we have

$$\sup(A \cup B) = \max(\sup A, \sup B),$$
$$\inf(A \cup B) = \min(\inf A, \inf B).$$

3) If $A \cap B \neq \emptyset$, then $A \cap B$ is bounded and we have

$$\sup(A \cap B) \leq \min(\sup A, \sup B),$$
$$\inf(A \cap B) \geq \max(\inf A, \inf B).$$

4) Let $A + B = \{x + y \ / \ x \in A \wedge y \in B\}$, then $A + B$ is bounded and nonempty and we have

$$\sup(A + B) = \sup A + \sup B,$$
$$\inf(A + B) = \inf A + \inf B.$$

5) Let $-A = \{-x \ / \ x \in A\}$, then $-A$ is bounded and nonempty and we have

$$\sup(-A) = -\inf A,$$
$$\inf(-A) = -\sup A.$$

### 1.1.15 Archimedes' axiom

$$\forall x \in \mathbb{R}, \forall y > 0, \exists n \in \mathbb{N}^* \ / \ n.y > x.$$

**Example 30** *let $x = 5$ and $y = 2$. Find $n \in \mathbb{N}^*$ such that $n.y > x$,*
$n > \dfrac{x}{y} = \dfrac{5}{2} \Longrightarrow n > 2,5$. *Just take $n = 3$.*

### 1.1.16 The greatest integer function (Floor function)

**Definition 31** *Let $x \in \mathbb{R}$. The greatest integer of $x$ is the greatest integer less than or equal to $x$. It is noted $E(x)$ or $[x]$ or $\lfloor x \rfloor$.*
$\forall x \in \mathbb{R}, \exists n \in \mathbb{Z} \ / \ n \leq x < n+1.$
$[.] : \mathbb{R} \longrightarrow \mathbb{Z} \ / \ [x] = E(x) = n.$

**Example 32** $E(2,7) = [2,7] = 2 \qquad (2 \leq 2,7 < 3).$
$E(-3,1) = [-3,1] = -4 \qquad (-4 \leq -3,1 < -3).$

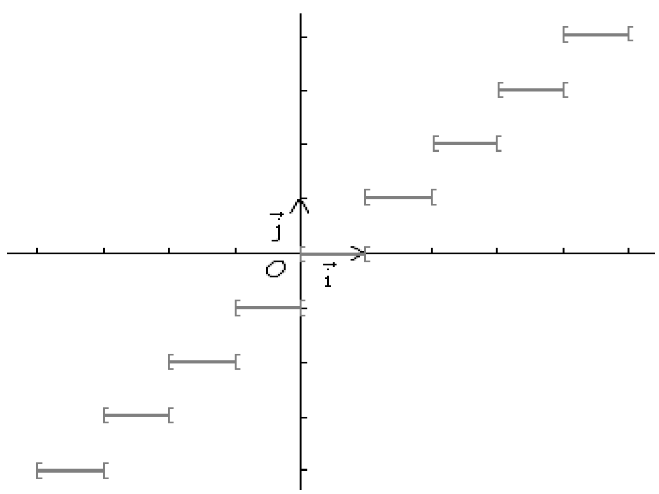
**Property**

$$\forall x \in \mathbb{R}, \ \ [x] \leq x < [x] + 1,$$

then

$$\forall x \in \mathbb{R}, \ x - 1 < [x] \leq x.$$

**Graph of the greatest integer function**



### 1.1.17 Density of $\mathbb{Q}$ in $\mathbb{R}$

**Theorem 33** ***(Density theorem)***
*Let $x, y \in \mathbb{R}$ be such that $x < y$, then $\exists q \in \mathbb{Q}$ / $x < q < y$.*

**Conclusion 34** *Between two distinct real numbers, there exists a rational.*
*So between two distinct real numbers, there exists an infinity of rational numbers.*

### 1.1.18 Notion of topology in $\mathbb{R}$

**Definition 35** *(Adherent point )*
*Let $A$ be a nonempty part of $\mathbb{R}$. We say that $x$ is an adherent point of $A$ if every open interval $I$ containing $x$ meets $A$ (i.e. $I \cap A \neq \emptyset$). The set of adherent points of $A$ is denoted $\overline{A}$: the closure of $A$.*
*$x \in \overline{A} \Longleftrightarrow \forall I$ open interval containing $x$, $I \cap A \neq \emptyset$.*

**Remark 36** 1) $A \subset \overline{A}$.
2) $x \in \overline{A} \Longleftrightarrow \forall \varepsilon > 0, ]x - \varepsilon, x + \varepsilon[ \cap A \neq \emptyset$.

**Example 37** 1) $A = [0, 1] \Longrightarrow \overline{A} = [0, 1]$.
2) $A = ]-1, 1[ \Longrightarrow \overline{A} = [-1, 1]$.
3) $A = \left\{ \frac{1}{n} \ / \ n \in \mathbb{N}^* \right\} \Longrightarrow \overline{A} = \left\{ \frac{1}{n} \ / \ n \in \mathbb{N}^* \right\} \cup \{0\}$.

### 1.1.19 Application

**Exercise 38** *Let the set* $A = \left\{ \frac{n+1}{n-2} \ / \ \ n \in \mathbb{N}, n \geq 3 \right\}$.
1) *Prove that $A$ is nonempty and bounded.*
2) *Using the characterization of the supremum and infimum, prove that* $\sup A = 4$ *and* $\inf A = 1$.
3) *Determine* $\max A$ *and* $\min A$ *if they exist.*
4) *Deduce the supremum and the infimum of the following sets :*

$$B = \left\{ \frac{n+1}{n-2} + \frac{1}{2} \ / \ \ n \in \mathbb{N}, n \geq 3 \right\},$$

$$D = \left\{ \frac{n+1}{n-2} \ , \ (-1)^n \ / \ n \in \mathbb{N}, n \geq 3 \right\}.$$

**Solution :**

1) For $n = 3$, $x_3 = \dfrac{3+1}{3-2} = 4 \in A$, then $A$ is nonempty.

Now, we will prove that $A$ is bounded.

We first decompose the fraction $\dfrac{n+1}{n-2}$ in the following form:

$$\frac{n+1}{n-2} = a + \frac{b}{n-2},$$

$$\frac{n+1}{n-2} = \frac{a(n-2)+b}{n-2} = \frac{an-2a+b}{n-2},$$

by identification, we obtain

$$\begin{cases} a & = & 1 \\ -2a+b & = & 1 \end{cases} \Longrightarrow \begin{cases} a & = & 1, \\ b & = & 3, \end{cases}$$

then

$$\frac{n+1}{n-2} = 1 + \frac{3}{n-2}.$$

Using this decomposition, we will show that $A$ is bounded.

$$n \geq 3 \Longrightarrow n-2 \geq 1 \Longrightarrow 0 \leq \frac{1}{n-2} \leq 1 \Longrightarrow 0 \leq \frac{3}{n-2} \leq 3,$$

so

$$1 \leq 1 + \frac{3}{n-2} \leq 4 \Longrightarrow 1 \leq \frac{n+1}{n-2} \leq 4, \tag{1.1}$$

hence, $A$ is bounded.

2) $A$ is nonempty and bounded, so according to the axiom of the supremum and

the axiom of the infimum, $\sup A$ and $\inf A$ exist.

Now, we will prove that $\sup A = 4$ :

From (1.1) 4 is an upper bound of $A$,

since $4 \in A$ and for all $\varepsilon > 0$ we have $4 > 4 - \varepsilon$,

so, from the characterization of the supremum, we deduce that $\sup A = 4$.

We will prove that $\inf A = 1$ :

We check if $1 \in A$ :

We suppose that $1 \in A \Longrightarrow \exists n \in \mathbb{N}, n \geq 3 \ / \ \dfrac{n+1}{n-2} = 1$

$\Longrightarrow n+1 = n-2 \Longrightarrow 1 = -2$ : absurd, then $1 \notin A$.

We will prove that $\inf A = 1$, using the characterization of the infimum :

$$\inf A = 1 \Longleftrightarrow \begin{cases} 1°) \ \forall x \in A, \ \ x \geq 1, \\ 2°) \ \forall \varepsilon > 0, \exists x_0 \in A \ / \ x_0 < 1 + \varepsilon, \end{cases}$$

or equivalently :

$$\inf A = 1 \Longleftrightarrow \begin{cases} 1^\circ)\ \forall n \in \mathbb{N}, n \geq 3,\ \dfrac{n+1}{n-2} \geq 1, \\ 2^\circ)\ \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, n_0 \geq 3\ \ /\ \ \dfrac{n_0+1}{n_0-2} < 1+\varepsilon. \end{cases}$$

From (1.1) the condition $1^\circ)$ is verified.
Now, we wille prove the condition $2^\circ)$ : let $\varepsilon > 0$,

$$\frac{n_0+1}{n_0-2} < 1+\varepsilon \Longleftrightarrow 1+\frac{3}{n_0-2} < 1+\varepsilon >\Longleftrightarrow \frac{3}{n_0-2} < \varepsilon \Longleftrightarrow n_0 > \frac{3}{\varepsilon}+2.$$

Just take $n_0 = \left[\dfrac{3}{\varepsilon}+2\right]+1 = \left[\dfrac{3}{\varepsilon}\right]+3 \geq 3$,
therefore, $\inf A = 1$.

3) $\sup A = 4 \in A \Longrightarrow \max A = 4$.
$\inf A = 1 \notin A \Longrightarrow \min A$ does not exist.

4) $B = \left\{\dfrac{n+1}{n-2}+\dfrac{1}{2}\ /\ n \in \mathbb{N}, n \geq 3\right\} = \left\{\dfrac{n+1}{n-2}\ /\ n \in \mathbb{N}, n \geq 3\right\} + \left\{\dfrac{1}{2}\right\}$,

then, $B = A + \left\{\dfrac{1}{2}\right\}$.

By the properties of the supremum and the infimum, $B$ is nonempty and bounded, $\sup B$ et $\inf B$ exist and in addition we have

$$\sup B = \sup\left(A+\left\{\frac{1}{2}\right\}\right) = \sup A + \sup\left\{\frac{1}{2}\right\} = 4+\frac{1}{2} = \frac{9}{2},$$
$$\inf B = \inf\left(A+\left\{\frac{1}{2}\right\}\right) = \inf A + \inf\left\{\frac{1}{2}\right\} = 1+\frac{1}{2} = \frac{3}{2}.$$

$$D = \left\{\frac{n+1}{n-2}\ ,\ (-1)^n\ /\ n \in \mathbb{N}, n \geq 3\right\},$$

$D = \left\{\dfrac{n+1}{n-2}\ /\ n \in \mathbb{N}, n \geq 3\right\} \cup \{(-1)^n\ /\ n \in \mathbb{N}, n \geq 3\}$,
then, $D = A \cup \{-1, 1\}$.
By the properties of the supremum and the infimum, $D$ is nonempty and bounded, $\sup D$ et $\inf D$ exist and in addition we have

$$\sup D = \sup(A \cup \{-1,1\}) = \max(\sup A, \sup\{-1,1\}) = \max(4,1) = 4,$$
$$\inf D = \inf(A \cup \{-1,1\}) = \min(\inf A, \inf\{-1,1\}) = \min(1,-1) = -1.$$

## 1.2 The set of complex numbers

### 1.2.1 Introduction

In $\mathbb{R}$, a negative number cannot be the square of a real number. We construct the set $\mathbb{C}$, called the set of complex numbers, containing $\mathbb{R}$, which equipped with the laws "+" and "×" is a commutative field and such that any element can be a square in $\mathbb{C}$ (even negative real numbers).

**Definition 39** *A complex number is an ordered pair $z = (x, y)$ such that $x, y \in \mathbb{R}$.*

*We write $z = x + iy$ where $i^2 = -1$.*

*$i = (0, 1)$ : is the pure imaginary number,*

*$\operatorname{Re} z = x$ : is the real part of $z$,*

*$\operatorname{Im} z = y$ : is the imaginary part of $z$.*

**Remark 40** *- If $y = 0$, then $z = x$ is a real number.*

*- If $x = 0$, then $z = iy$ is a pure imaginary number.*

### 1.2.2 Operations on complex numbers

Let $z = x + iy$ and $z' = x' + iy' \in \mathbb{C}$.

We define on $\mathbb{C}$, the addition by : $z + z' = x + x' + i(y + y')$,

and the multiplication by : $z.z' = (x+iy)(x'+iy') = (xx'-yy')+i(xy'+x'y)$.

$\mathbb{C}$, equipped with these two operations is a commutative field.

**Remark 41** *If $z = x + iy \neq 0$, then*

$$\frac{1}{z} = \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2}.$$

### 1.2.3 The conjugate of a complex number

Let $z = x + iy \in \mathbb{C}$.

The conjugate of $z$ is the complex number $\overline{z} = x - iy$.

**Properties :**

1)$\forall z_1, z_2 \in \mathbb{C}$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

2)$\forall z_1, z_2 \in \mathbb{C}$, $\overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$.

### 1.2.4 The modulus of a complex number

The modulus of a complex number $z = x+iy$ is a function from $\mathbb{C}$ to $\mathbb{R}_+$ defined as follows

$$\begin{array}{rccl} |.| : & \mathbb{C} & \longrightarrow & \mathbb{R}_+ \\ & z & \longrightarrow & |z| = \sqrt{z.\overline{z}} = \sqrt{x^2+y^2}. \end{array}$$

**Remark 42** *The modulus coincides with the absolute value when $z$ is real.*

**Properties of modulus** :
$\forall z, z_1, z_2 \in \mathbb{C}$, we have the following properties :
1) $|z| = 0 \Longleftrightarrow z = 0$.
2) $|z_1.z_2| = |z_1| . |z_2|$.
3) $|z_1 + z_2| \leq |z_1| + |z_2|$.
4) $\forall z \neq 0, \left|\frac{1}{z}\right| = \frac{1}{|z|}$.

**Remark 43** *We cannot order the set of complex numbers. So we cannot compare or write inequalities between two complex numbers.*

### 1.2.5 Euler's formula

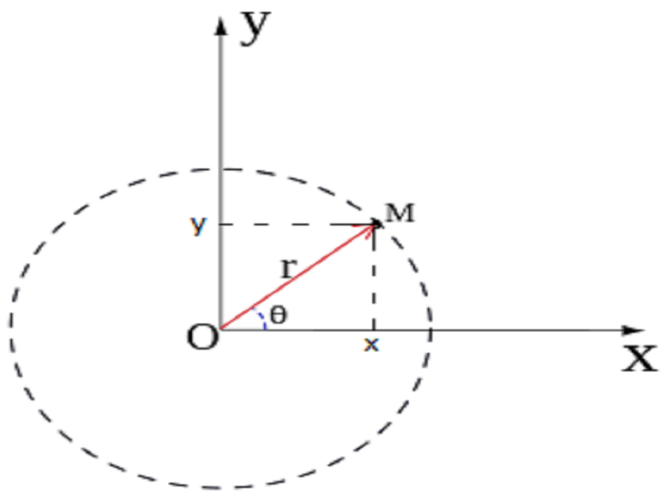
$e^{i\theta} = \cos\theta + i\sin\theta, \;\; \theta \in \mathbb{R}$.

### 1.2.6 The complex exponential function

Let $z = x + iy \in \mathbb{C}$.
We set by definition

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

**Properties :**
1) $\forall z_1, z_2 \in \mathbb{C}, e^{z_1+z_2} = e^{z_1}.e^{z_2}$.
2) $\forall z \in \mathbb{C}, e^z \neq 0 \quad (e^z.e^{-z} = e^{z-.z} = e^0 = 1)$.




### 1.2.7 Trigonometric form of a complex number

Let $z = x + iy \in \mathbb{C}$ such that $z \neq 0$,
$|z| = \sqrt{x^2 + y^2} = r$.

The point M with coordinates $(x, y)$ is called the image of the complex number $z$.
$z$ is the affix of the point $M$.

$$\begin{aligned} \cos\theta &= \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{|z|}, \\ \sin\theta &= \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{|z|}. \end{aligned}$$

The number $\theta$ is defined to within $2\pi$.
$\theta$ is called the argument of z and is denoted $\arg z$.
Thus

$$\begin{aligned} x &= |z| \cos\theta, \\ y &= |z| \sin\theta. \end{aligned}$$

Therefore

$$z = x + iy = |z| \cos\theta + i\,|z| \sin\theta = |z|\,(\cos\theta + i\sin\theta) = |z|\, e^{i\theta}.$$

If $z = 0$, so we have $r = |z| = 0$ and $\theta$ any argument.

**Conclusion :**
$\forall z \in \mathbb{C}, \exists (r, \theta) \in \mathbb{R}_+ \times \mathbb{R} \;/\; z = r(\cos\theta + i\sin\theta) = re^{i\theta}$,
$\theta := \arg z$ and $r = |z|$.

This representation is very useful for multiplication and division of complex numbers.
Indeed, $\forall z_1, z_2 \in \mathbb{C}$, we have

$$z_1.z_2 = r_1 e^{i\theta_1}.r_2 e^{i\theta_2} = r_1.r_2 e^{i(\theta_1+\theta_2)},$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

### 1.2.8 Moivre's formula

$\left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i \sin n\theta, \forall \theta \in \mathbb{R}, \forall n \in \mathbb{N}.$

**Remark 44** *We have*
$e^{i\theta} = \cos\theta + i \sin\theta, \quad \theta \in \mathbb{R},$
$e^{-i\theta} = \cos\theta - i \sin\theta,$
*from which we deduce the following formulas :*

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \theta \in \mathbb{R}, \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

## 1.3 Exercises

**Exercise 45** *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function on $\mathbb{R}$ and $A$ a non-empty and bounded above subset of $\mathbb{R}$. We assume that $f(A)$ is bounded above.*
*Prove that* $\sup f(A) \leq f(\sup A)$.

**Solution** :
Since $A$ and $f(A)$ are non-empty and bounded above subsets of $\mathbb{R}$, then, according to the supremum axiom, $\sup A$ and $\sup f(A)$ exist. On the other hand, we have $\forall x \in A$, $x \leq \sup A$.
Since the function $f$ is increasing, we obtain

$$\forall x \in A, f(x) \leq f(\sup A),$$

then, $f(\sup A)$ is an upper bound of $f(A)$ and since $\sup f(A)$ is the smallest upper bound of $f(A)$,
then we deduce that $\sup f(A) \leq f(\sup A)$.

**Exercise 46** *Let the set* $A = \left\{ \frac{1}{n} + \frac{1}{n^2}, n \in \mathbb{N}^* \right\}$.

**1**) *Prove that* $A$ *is non-empty and bounded.*
**2**) *Using the characterization of the supremum and infimum, prove that* $\sup A = 2$ *and* $\inf A = 0$.
**3**) *Determine* $\max A$ *and* $\min A$ *if they exist.*
4) *Deduce the supremum and infimum of the set.*
$B = \left\{ \frac{n+1}{n^2} + 2 \ / \ n \in \mathbb{N}^* \right\}$.

**Solution :**
Let the set $A = \left\{ \frac{1}{n} + \frac{1}{n^2}, n \in \mathbb{N}^* \right\}$.
**1**) We prove that $A$ is non-empty and bounded.
For $n = 1$, $x_1 = 2 \in A$, then $A \neq \emptyset$.
We prove that $A$ is bounded. We have
$\forall n \in \mathbb{N}^*, \quad 0 \leq \frac{1}{n} \leq 1,$
$\forall n \in \mathbb{N}^*, \quad 0 \leq \frac{1}{n^2} \leq 1,$
then, $\forall n \in \mathbb{N}^*, \quad 0 \leq \frac{1}{n} + \frac{1}{n^2} \leq 2 ......(*)$
hence, $A$ is bounded.

**2**) We prove that $\sup A = 2$ and $\inf A = 0$.

$A$ is a subset of $\mathbb{R}$ non-empty and bounded, thus, according to the supremum and infimum axioms, $\sup A$ et $\inf A$ exist.

Now, we prove that $\sup A = 2$.

From $(*)$, 2 is an upper bound of $A$.

Moreover, we have: $\forall \varepsilon > 0, \exists n = 1 \in \mathbb{N}^* \ / \ \frac{1}{n} + \frac{1}{n^2} = 2 > 2 - \varepsilon,$

then, according to the characterization of the supremum, $\sup A = 2$.

We prove that $\inf A = 0$ : we use the characterization of the infimum.

$$\inf A = 0 \Longleftrightarrow \begin{cases} 1°) \forall n \in \mathbb{N}^*, \frac{1}{n} + \frac{1}{n^2} \geq 0, \\ 2°) \forall \varepsilon > 0, \exists n \in \mathbb{N}^* \ / \ \frac{1}{n} + \frac{1}{n^2} < 0 + \varepsilon. \end{cases}$$

1°) It is satisfied according to $(*)$.
2°) Let $\varepsilon > 0$, we have $\frac{1}{n} + \frac{1}{n^2} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n},$
so it is enough that $\frac{2}{n} < \varepsilon,$

$$\frac{2}{n} < \varepsilon \Longleftrightarrow n > \frac{2}{\varepsilon},$$

it is enough to take $n = \left[\frac{2}{\varepsilon}\right] + 1$, then $\inf A = 0$.

**3**) We determine $\max A$ and $\min A$ if they exist.

$\sup A = 2 \in A \Longrightarrow \max A = 2$.

$\inf A = 0 \notin A$ (otherwise $\exists n \in \mathbb{N}^* \ / \ \frac{1}{n} + \frac{1}{n^2} = \frac{n+1}{n^2} = 0 \Longrightarrow n = -1$: absurd), then $\min A$ does not exist.

4) We deduce the supremum and infimum of the set
$B = \left\{ \frac{n+1}{n^2} + 2 \ / \ n \in \mathbb{N}^* \right\}$.

$B = A + \{2\}$ : it is a bounded and non-empty set,

$\sup B = \sup A + \sup \{2\} = 2 + 2 = 4$,

$\inf B = \inf A + \inf \{2\}) = 0 + 2 = 2$.

**Exercise 47** *Let the set* $A = \left\{ \frac{3n+2}{n+4}, n \in \mathbb{N} \right\}$.
**1**) *Prove that* $A$ *s non-empty and bounded.*
**2**) *Using the characterization of the supremum and infimum, show that* $\sup A = 3$ *and* $\inf A = \frac{1}{2}$.
**3**) *Determine* $\max A$ *and* $\min A$ *if they exist.*
**4**) *Deduce the supremum and infimum of the set*
$B = \left\{ \frac{3n+2}{n+4}, (-1)^n \ / \ n \in \mathbb{N} \right\}$.

**Solution :**
Let the set $A = \left\{ \frac{3n+2}{n+4}, n \in \mathbb{N} \right\}$.

**1**) We prove that $A$ is non-empty and bounded.
Pour $n = 0$, $x_0 = \frac{1}{2} \in A$, then $A$ is non-empty.
Now, we prove that $A$ is bounded:

$\frac{3n+2}{n+4} = a + \frac{b}{n+4} = 3 - \frac{10}{n+4}$.

$n \in \mathbb{N} \Longrightarrow n \geq 0 \Longrightarrow n + 4 \geq 4 \Longrightarrow 0 \leq \frac{1}{n+4} \leq \frac{1}{4}$,

$\frac{1}{2} \leq 3 - \frac{10}{n+4} \leq 3 \Longrightarrow \frac{1}{2} \leq \frac{3n+2}{n+4} \leq 3 ......(*)$,

then, $A$ is bounded.

**2**) Using the characterization of the supremum and infimum, show that $\sup A = 3$ et $\inf A = \dfrac{1}{2}$.

Since $A \neq \emptyset$ and bounded, thus according to the supremum and infimum axioms, $\sup A$ et $\inf A$ exist.
We prove that $\inf A = \dfrac{1}{2}$:
From $(*)$, $\dfrac{1}{2}$ is a lower bound of $A$.
Moreover $\forall \varepsilon > 0, \exists n = 0 \in \mathbb{N} \ / \ \dfrac{3n+2}{n+4} = \dfrac{1}{2} < \dfrac{1}{2} + \varepsilon$.
Thus, according to the characterization of the infimum, $\inf A = \dfrac{1}{2}$.

We prove that $\sup A = 3$ using the characterization of the supremum

$$\sup A = 3 \Longleftrightarrow \begin{cases} 1°)\ \forall n \in \mathbb{N}, \dfrac{3n+2}{n+4} \leq 3, \\ 2°)\ \forall \varepsilon > 0, \exists n \in \mathbb{N} \ / \ \dfrac{3n+2}{n+4} > 3 - \varepsilon. \end{cases}$$

1°) Already proven in the first question.

2°) Let $\varepsilon > 0$,

$\dfrac{3n+2}{n+4} > 3 - \varepsilon \Longleftrightarrow n > \dfrac{10}{\varepsilon} - 4$,

it is enough to take $n = \left[\left|\dfrac{10}{\varepsilon} - 4\right|\right] + 1 \in \mathbb{N}$.

**3**) Determine $\max A$ and $\min A$ if they exist.

$\inf A = \dfrac{1}{2} \in A$, then $\min A = \dfrac{1}{2}$.

$\sup A = 3 \notin A$ (otherwise $3 \in A \Longleftrightarrow \dfrac{3n+2}{n+4} = 3 \Longleftrightarrow 2 = 12$ : absurd), thus $\max A$ does not exist.

**4**) Deduce the supremum and infimum of the set :
$B = \left\{\dfrac{3n+2}{n+4}, (-1)^n \ / \ n \in \mathbb{N}\right\}$.

$B = A \cup \{-1, 1\}$ : it is a bounded and non-empty set.

$\sup B = \max(\sup A, \sup\{-1, 1\}) = \max(3, 1) = 3$,

$\inf B = \min(\inf A, \inf\{-1, 1\}) = \min(\dfrac{1}{2}, -1) = -1$.

**Exercise 48** 1) *Determine the general form of the solutions of the equation :*

$$z^n = a \ , \ \ n \in \mathbb{N}^* \ , \ a, z \in \mathbb{C}$$

2) *Find all solutions to the equation* $z^3 = 1$.

**Solution :**
1) $z^n = a \ , \ \ n \in \mathbb{N}^* \ , \ a, z \in \mathbb{C}$.

Complex numbers are written in trigonometric form. :
$z = \rho e^{i\theta}, \ \ \rho = |z|$ and $\theta = \arg z$,

$a = re^{i\alpha}, \ \ r = |a|$ and $\alpha = \arg a$.

$z^n = a \Longleftrightarrow (\rho e^{i\theta})^n = re^{i\alpha} \Longleftrightarrow \rho^n e^{in\theta} = re^{i\alpha}$,

then, $\rho^n = r$ and $n\theta = \alpha + 2k\pi$,

hence, $\rho = \sqrt[n]{r}$ and $\theta = \dfrac{\alpha + 2k\pi}{n}, \quad k = 0, 1, ...., (n-1)$.

Therefore, the equation admits $n$ solutions :

$$S = \left\{ z_k = \sqrt[n]{r} e^{i\frac{\alpha+2k\pi}{n}}, \quad k = 0, 1, ...., (n-1) \right\}.$$

2) We are looking for all the solutions of the equation $z^3 = 1$.

$1 = 1e^{i0} \Longrightarrow r = 1$ and $\alpha = 0$,

$z_k = e^{i\frac{2k\pi}{3}}, \quad k = 0, 1, 2.$

then, the equation $z^3 = 1$ admits three solutions :

$z_0 = e^{i0} = 1$,

$$z_1 = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$z_2 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

**Exercise 49** 1) *Find the solutions to the equation :* $z^3 = 1 - i\sqrt{3}$.

2) *Solve in* $\mathbb{C}$ *the following equation :* $z^2 - 2iz + 1 - i = 0$.

**Solution :**
1) The solutions of $z^3 = 1 - i\sqrt{3}$ :

$z^3 = 1 - i\sqrt{3} \Longleftrightarrow \rho^3 e^{i3\theta} = 2e^{-i\frac{\pi}{3}}$,

then, $\rho^3 = 2$ and $3\theta = -\dfrac{\pi}{3} + 2k\pi$, hence, $\rho = \sqrt[3]{2}$ and $\theta = \dfrac{-\frac{\pi}{3} + 2k\pi}{3} = -\dfrac{\pi}{9} + \dfrac{2k\pi}{3}, \quad k = 0, 1, 2.$

$S = \left\{ z_k = \sqrt[3]{2} e^{i\left(-\frac{\pi}{9} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2. \right\}.$

Therefore, the equation $z^3 = 1 - i\sqrt{3}$ admits 3 solutions :

$z_0 = \sqrt[3]{2} e^{i\left(-\frac{\pi}{9}\right)} = \sqrt[3]{2}\left(\cos\dfrac{-\pi}{9} + i \sin\dfrac{-\pi}{9}\right),$

$z_1 = \sqrt[3]{2} e^{i\left(-\frac{\pi}{9} + \frac{2\pi}{3}\right)} = \sqrt[3]{2}\left(\cos\dfrac{5\pi}{9} + i \sin\dfrac{5\pi}{9}\right),$

$z_2 = \sqrt[3]{2} e^{i\left(-\frac{\pi}{9} + \frac{4\pi}{3}\right)} = \sqrt[3]{2}\left(\cos\dfrac{11\pi}{9} + i \sin\dfrac{11\pi}{9}\right).$

2) We solve the following equation in $\mathbb{C}$ : $z^2 - 2iz + 1 - i = 0.$

$\Delta = -8 + 4i = \omega^2 = (x + iy)^2 = x^2 - y^2 + 2ixy,$

$|-8 + 4i| = |x + iy|^2 \Longleftrightarrow x^2 + y^2 = \sqrt{80} = 4\sqrt{5},$

by identification, we obtain :

$$\left\{ \begin{array}{ccc} x^2 - y^2 & = & -8 \\ x^2 + y^2 & = & 4\sqrt{5} \\ 2xy & = & 4 \end{array} \right. \Longrightarrow \left\{ \begin{array}{cc} x^2 = & -4 + 2\sqrt{5} \\ y^2 = & 4 + 2\sqrt{5} \end{array} \right. \Longrightarrow \left\{ \begin{array}{cc} x = & \pm\sqrt{-4 + 2\sqrt{5}}, \\ y = & \pm\sqrt{4 + 2\sqrt{5}}. \end{array} \right.$$

Since $xy = 2$, then $x$ and $y$ are of the same sign,

then, $\omega = \sqrt{-4 + 2\sqrt{5}} + i\sqrt{4 + 2\sqrt{5}},$

or $\omega = -\sqrt{-4 + 2\sqrt{5}} - i\sqrt{4 + 2\sqrt{5}}.$

We choose $\omega = \sqrt{-4 + 2\sqrt{5}} + i\sqrt{4 + 2\sqrt{5}},$

therefore, $z_1 = \dfrac{2i + \omega}{2}$ and $z_2 = \dfrac{2i - \omega}{2}.$

# Chapter 2

# The numerical sequences

## 2.1 Introduction

**Definition 50** *A sequence is a function from* $\mathbb{N}$ *to* $\mathbb{R}$

$$\begin{array}{rccll} U: & \mathbb{N} & \longrightarrow & \mathbb{R} & \\ & n & \longrightarrow & U(n) & = U_n. \end{array}$$

*We note the sequence of general term* $U_n$ *by* $(U_n)_n$.

**Remark 51** $(U_n)_n = (V_n)_n \Longleftrightarrow (\forall n \in \mathbb{N},\ U_n = V_n)$.

### 2.1.1 Sequences given as a function of n

A sequence can be given as a function of $n$.

**Example 52** *:* $U_n = \dfrac{n+1}{2n+3}, \quad n \in \mathbb{N},$

$U_0 = \dfrac{1}{3},\ U_1 = \dfrac{2}{5},\ U_2 = \dfrac{3}{7}, .......$

### 2.1.2 Recursive sequences

Recursive sequences are sequences where each term is defined based on previous terms. We can define a recursive sequence by giving the first term $U_0 = \alpha$ and the recurrence relation $U_{n+1} = f(U_n)$ for all $n \in \mathbb{N}$, where $f$ is a function from $D \subseteq \mathbb{R}$ in $\mathbb{R}$ assuming, of course, that $f(D) \subseteq D$ for the sequence to be well-defined.

**Example 53**

$$\begin{cases} U_0 & = & 1, \\ U_{n+1} & = & \dfrac{U_n+2}{2U_n+3}, \quad \forall n \in \mathbb{N}. \end{cases}$$

$$U_{n+1} = f(U_n) = \frac{U_n+2}{2U_n+3},\ avec\ f : D = [0, +\infty[ \longrightarrow \mathbb{R}/f(x) = \frac{x+2}{2x+3}$$

$$U_1 = \frac{U_0+2}{2U_0+3} = \frac{3}{5}, .....$$

### 2.1.3 Operations on sequences

The addition : $(U_n)_n + (V_n)_n = (U_n + V_n)_n$.

The multiplication : $(U_n).(V_n) = (U_n.V_n)_n$.

We note $F(\mathbb{N}, \mathbb{R})$ the set of numerical sequences.

$F(\mathbb{N}, \mathbb{R})$ equipped with addition and multiplication is a unital commutative ring.

## 2.2 The different types of sequences

### 2.2.1 Monotonous sequences

- A sequence $(U_n)_n$ is increasing $\Longleftrightarrow (\forall n \in \mathbb{N},\ U_{n+1} \geq U_n)$.
- A sequences $(U_n)_n$ is strictly increasing $\Longleftrightarrow (\forall n \in \mathbb{N},\ U_{n+1} > U_n)$.
- A sequence $(U_n)_n$ in decreasing $\Longleftrightarrow (\forall n \in \mathbb{N},\ U_{n+1} \leq U_n)$.
- A sequence $(U_n)_n$ is strictly decreasing $\Longleftrightarrow (\forall n \in \mathbb{N},\ U_{n+1} < U_n)$.
- A sequence $(U_n)_n$ is monotonous $\Longleftrightarrow (U_n)_n$ is increasing or decreasing.
- A sequence $(U_n)_n$ is constant or stationary $\Longleftrightarrow (\forall n \in \mathbb{N},\ U_{n+1} = U_n)$.

### 2.2.2 The bounded sequences

- A sequence $(U_n)_n$ is bounded above $\Longleftrightarrow (\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, U_n \leq M)$.
- A sequence $(U_n)_n$ is bounded below $\Longleftrightarrow (\exists m \in \mathbb{R}, \forall n \in \mathbb{N}, U_n \geq m)$.
- A sequence $(U_n)_n$ is bounded $\Longleftrightarrow (U_n)_n$ is bounded above and bounded below.

$$\Longleftrightarrow (\exists M, m \in \mathbb{R}, \forall n \in \mathbb{N}, m \leq U_n \leq M).$$

$$\Longleftrightarrow (\exists \alpha > 0, \forall n \in \mathbb{N}, |U_n| \leq \alpha).$$

$B(\mathbb{N}, \mathbb{R})$ is the set of bounded sequences, it is a subring of $F(\mathbb{N}, \mathbb{R})$.

**Example 54** *Study the monotony of the sequence of general term* $U_n = \dfrac{n-1}{3n+2}$, $n \in \mathbb{N}$.

$$U_{n+1} - U_n = \frac{n}{3n+5} - \frac{n-1}{3n+2} = \frac{5}{(3n+5)(3n+2)} > 0,$$

*then,* $(U_n)_n$ *is increasing.*

**Example 55** *Study the monotony of the sequence of general term*

$U_n = \dfrac{1}{n}, \quad n \in \mathbb{N}^*.$

$$U_{n+1} - U_n = \frac{1}{n+1} - \frac{1}{n} = \frac{-1}{(n+1)n} < 0,$$

*hence,* $(U_n)_n$ *is decreasing.*

**Example 56** *Prove that the sequence of general term* $U_n = \dfrac{n}{2n+1}$, $n \in \mathbb{N}$ *is bounded.*

$$\forall n \in \mathbb{N}, 2n \leq 2n+1 \Longrightarrow \frac{1}{2n+1} \leq \frac{1}{2n} \Longrightarrow \frac{n}{2n+1} \leq \frac{n}{2n} = \frac{1}{2},$$

*then,* $0 \leq \dfrac{n}{2n+1} \leq \dfrac{1}{2}$,

*therefore,* $(U_n)_n$ *is bounded.*

**Example 57** *Let the sequence of general term* $U_n = (-1)^n \quad n \in \mathbb{N}$.

*This sequence is neither increasing nor decreasing, it is an alternating sequence, but it is bounded because we have* $\forall n \in \mathbb{N},\ -1 \leq U_n \leq 1$.

## 2.3 The nature of sequences

### 2.3.1 Convergent sequences

**Definition 58** *We say that the sequence* $(U_n)_n$ *is convergent if there exists* $\ell \in \mathbb{R}$ *such that*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n \geq N \Longrightarrow |U_n - \ell| < \varepsilon).$$

*We say that the sequence* $(U_n)_n$ *converges to the limit* $\ell$ *and we write :* $\lim\limits_{n \longrightarrow +\infty} U_n = \ell$.

**Example 59** *Using the definition, show that* $\lim\limits_{n\longrightarrow+\infty}\frac{1}{n}=0$.

$$\lim_{n\longrightarrow+\infty}\frac{1}{n}=0\Longleftrightarrow(\forall\varepsilon>0,\exists N\in\mathbb{N}^*,\forall n\in\mathbb{N}^*,(n\geq N\Longrightarrow\left|\frac{1}{n}-0\right|<\varepsilon)).$$

$$\left|\frac{1}{n}\right|<\varepsilon\Longleftrightarrow\frac{1}{n}<\varepsilon\Longleftrightarrow n>\frac{1}{\varepsilon},$$

*which means that* $n\geq\left[\frac{1}{\varepsilon}\right]+1=N$,

*then, it is sufficient to take* $N=\left[\frac{1}{\varepsilon}\right]+1\in\mathbb{N}^*$.

### 2.3.2 Infinite limits

$$\lim_{n\longrightarrow+\infty}U_n=+\infty\Longleftrightarrow(\forall A>0,\exists N\in\mathbb{N},\forall n\in\mathbb{N},(n\geq N\Longrightarrow U_n>A)).$$

$$\lim_{n\longrightarrow+\infty}U_n=-\infty\Longleftrightarrow(\forall A>0,\exists N\in\mathbb{N},\forall n\in\mathbb{N},(n\geq N\Longrightarrow U_n<-A)).$$

**Example 60** 1) $\lim\limits_{n\longrightarrow+\infty}(2n+1)=+\infty$.

2) $\lim\limits_{n\longrightarrow+\infty}(-n+1)=-\infty$.

### 2.3.3 Divergent sequences

A divergent sequence is a sequence that does not converge, meaning a divergent sequence is one whose limit is infinite or does not exist.

**Example 61** *Let the sequence of general term* $U_n=(-1)^n$.

*we have* $U_{2n}=1$ *and* $U_{2n+1}=-1$, *then* $\lim\limits_{n\longrightarrow+\infty}U_n$ *does not exist.*

## 2.4 Main properties of sequences

**Theorem 62** *Every convergent numerical sequence has a unique limit.*

**Theorem 63** *Every convergent numerical sequence is bounded.*

**Remark 64** *The converse of the theorem is not true, that is, a bounded sequence is not necessarily convergent.*

**Example 65** *The sequence of general term $U_n = (-1)^n$ is bounded but it is not convergent.*

**Theorem 66** *Let $(U_n)_n$ be a bounded sequence and $(V_n)_n$ a sequence such that $\lim\limits_{n\longrightarrow+\infty} V_n = 0$, then $\lim\limits_{n\longrightarrow+\infty} U_n.V_n = 0$.*

**Example 67** $\lim\limits_{n\longrightarrow+\infty} \dfrac{\sin n}{n+1} = 0$ *since* $\sin n$ *is bounded and* $\lim\limits_{n\longrightarrow+\infty} \dfrac{1}{n+1} = 0$.

**Theorem 68** ***(Three Sequences Theorem)***

*Let $(V_n)_n$ and $(W_n)_n$ be two sequences that converge to the same limit $\ell$ and let $(U_n)_n$ a sequence such that for all $n \in \mathbb{N}, V_n \leq U_n \leq W_n$, then the sequence $(U_n)_n$ converges to $\ell$.*

**Example 69** *Prove that the sequence of general term $U_n = \dfrac{\cos n}{n^2+1}$ is convergent and calculate its limit.*

$\forall n \in \mathbb{N}, -1 \leq \cos n \leq 1 \Longrightarrow \dfrac{-1}{n^2+1} \leq \dfrac{\cos n}{n^2+1} \leq \dfrac{1}{n^2+1}$,

$\lim\limits_{n\longrightarrow+\infty} \dfrac{-1}{n^2+1} = \lim\limits_{n\longrightarrow+\infty} \dfrac{1}{n^2+1} = 0$, *so according to the three sequence theorem,* $\lim\limits_{n\longrightarrow+\infty} \dfrac{\cos n}{n^2+1} = 0$.

**Theorem 70** *Every real number is the limit of a sequence of rational numbers, i.e.*

$$\forall x \in \mathbb{R}, \exists (q_n) \subset \mathbb{Q} \ / \lim_{n\longrightarrow+\infty} q_n = x$$

Indeed, let $x \in \mathbb{R}, \forall n \in \mathbb{N}^*, x - \dfrac{1}{n} < x$,

according to the density theorem, there exists $q_n \in \mathbb{Q} \ / \ x - \dfrac{1}{n} < q_n < x$,

so according to the three sequence theorem we obtain $\lim\limits_{n\longrightarrow+\infty} q_n = x$.

**Theorem 71** - *Any increasing and bounded above sequence converges towards its spremum, i.e. $\lim\limits_{n\longrightarrow+\infty} U_n = \ell = \sup E$, où $E = \{U_n \in \mathbb{R}/n \in \mathbb{N}\}$.*

*- Any decreasing and bounded below sequence converges towards its infimum,, i.e. $\lim\limits_{n\longrightarrow+\infty} U_n = \ell = \inf E$.*

**Example 72** *Let the sequence of general term* $U_n = \dfrac{n-1}{3n+2}, \quad n \in \mathbb{N}$.

*It has already been shown that* $(U_n)_n$ *is increasing.*

*We also have* $\dfrac{n-1}{3n+2} \leq \dfrac{n}{3n} = \dfrac{1}{3}$, *then* $(U_n)_n$ *is bounded above.*

$(U_n)_n$ *being increasing and bounded above, therefore according to the previous theorem, it converges towards its supremum,* $\sup E = \lim\limits_{n \longrightarrow +\infty} U_n = \dfrac{1}{3}$.

**Theorem 73** *Let* $(U_n)_n$ *and* $(V_n)_n$ *be two sequences which converge respectively to* $\ell$ *and* $\ell'$, *then the sequences* $(U_n + V_n)_n, (\lambda U_n)_n, \lambda \in \mathbb{R}$ *and* $(U_n.V_n)_n$ *converge and we have*

1) $\lim\limits_{n \longrightarrow +\infty} (U_n + V_n) = \ell + \ell'$.

2) $\lim\limits_{n \longrightarrow +\infty} (\lambda U_n) = \lambda \ell$.

3) $\lim\limits_{n \longrightarrow +\infty} (U_n.V_n) = \ell.\ell'$.

4) *If* $\ell' \neq 0$ *and* $V_n \neq 0$, *then the sequence* $\left(\dfrac{U_n}{V_n}\right)_n$ *converges and* $\lim\limits_{n \longrightarrow +\infty} \dfrac{U_n}{V_n} = \dfrac{\ell}{\ell'}$.

**Theorem 74** *Let* $\ell \in \mathbb{R}$.

- *If* $\lim\limits_{n \longrightarrow +\infty} U_n = \ell$ *and* $\forall n \in \mathbb{N}, U_n \geq 0$, *then* $\ell \geq 0$.

- *If* $\lim\limits_{n \longrightarrow +\infty} U_n = \ell$ *and* $\forall n \in \mathbb{N}, U_n \leq 0$, *then* $\ell \leq 0$.

**Remark 75** - *If* $(U_n)_n$ *and* $(V_n)_n$ *are two convergente sequences such that for every* $n \in \mathbb{N}$, $U_n \leq V_n$, *then we have* $\lim\limits_{n \longrightarrow +\infty} U_n \leq \lim\limits_{n \longrightarrow +\infty} V_n$.

- *If* $\lim\limits_{n \longrightarrow +\infty} U_n = \ell$ *and* $\forall n \in \mathbb{N}, U_n > 0$, *then* $\ell \geq 0$.

**Proposition 76** *If* $\lim\limits_{n \longrightarrow +\infty} U_n = \ell$, *then* $\lim\limits_{n \longrightarrow +\infty} |U_n| = |\ell|$

*The converse is not true if* $\ell \neq 0$ *and it is true if* $\ell = 0$.

### 2.4.1 Arithmetic sequence and geometric sequence

**arithmetic sequence of common difference r :**

$U_0, U_1 = U_0 + r, U_1 = U_0 + 2r, ......., U_n = U_0 + nr.$

The partial sum of the terms of an arithmetic sequence:

$S_n = U_0 + U_1 + ...... + U_n = (U_0 + U_n)\dfrac{(n+1)}{2}$

**Geometric sequence with common ratio q :**

$U_0, U_1 = U_0 q, U_1 = U_0 q^2, ......., U_n = U_0 q^n.$

If $U_0 = 1$, then $U_n = q^n$ :

- If $-1 < q < 1$, then $\lim\limits_{n \longrightarrow +\infty} q^n = 0$,

- If $q = 1$, then $\lim\limits_{n \longrightarrow +\infty} q^n = 1$,

- If $q = -1$, then $\lim\limits_{n \longrightarrow +\infty} q^n$ does not exist,

- If $q > 1$, then $\lim\limits_{n \longrightarrow +\infty} q^n = +\infty$,

- If $q < -1$, then $\lim\limits_{n \longrightarrow +\infty} q^n$ does not exist.

**Conclusion :**

$(U_n)_n = (q^n)_n$ converges $\Longleftrightarrow q \in ]-1, 1]$.

The partial sum of the terms of a geometric sequence:

$S_n = U_0 + U_1 + ...... + U_n = U_0 \left(\dfrac{1 - q^{n+1}}{1 - q}\right)$, if $q \neq 1$.

If $q = 1$,then $S_n = U_0(n + 1)$.

### 2.4.2 Study of recursive sequences

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function, $U_0 \in D$ and $U_{n+1} = f(U_n), \forall n \in \mathbb{N}$.

We assume that $f(D) \subset D$ so that the sequence is well-defined.

If $f$ is increasing, then $(U_n)_n$ is monotonous :

- if $U_1 \geq U_0 \Longrightarrow (U_n)_n$ is increasing,
- if $U_1 \leq U_0 \Longrightarrow (U_n)_n$ is decreasing.

**Study of convergence**

We assume that $f$ is monotonous and continuous on $D$. If the sequence $(U_n)_n$ converges to $\ell$, then $\ell$ verifies the equation $\ell = f(\ell)$, so to find this limit, we just need to solve this equation.

### 2.4.3 Properties

1) $\lim\limits_{n\longrightarrow+\infty} U_n = +\infty, \forall n \in \mathbb{N}, U_n \leq V_n \Longrightarrow \lim\limits_{n\longrightarrow+\infty} V_n = +\infty.$

$\lim\limits_{n\longrightarrow+\infty} U_n = -\infty, \forall n \in \mathbb{N}, V_n \leq U_n \Longrightarrow \lim\limits_{n\longrightarrow+\infty} V_n = -\infty.$

2) $\lim\limits_{n\longrightarrow+\infty} U_n = +\infty, \lambda > 0 \Longrightarrow \lim\limits_{n\longrightarrow+\infty} \lambda U_n = +\infty.$

$\lim\limits_{n\longrightarrow+\infty} U_n = +\infty, \lambda < 0 \Longrightarrow \lim\limits_{n\longrightarrow+\infty} \lambda U_n = -\infty.$

3) $\lim\limits_{n\longrightarrow+\infty} U_n = \pm\infty, \forall n \in \mathbb{N}, U_n \neq 0 \Longrightarrow \lim\limits_{n\longrightarrow+\infty} \dfrac{1}{U_n} = 0.$

4) $\lim\limits_{n\longrightarrow+\infty} U_n = 0, \forall n \in \mathbb{N}, U_n > 0 \Longrightarrow \lim\limits_{n\longrightarrow+\infty} \dfrac{1}{U_n} = +\infty.$

5) $\lim\limits_{n\longrightarrow+\infty} U_n = +\infty \wedge \lim\limits_{n\longrightarrow+\infty} V_n = \ell \text{ (or } +\infty) \Longrightarrow \lim\limits_{n\longrightarrow+\infty} (U_n + V_n) = +\infty.$

6) $\lim\limits_{n\longrightarrow+\infty} U_n = +\infty \wedge \lim\limits_{n\longrightarrow+\infty} V_n = \ell > 0 \Longrightarrow \lim\limits_{n\longrightarrow+\infty} U_n.V_n = +\infty.$

**Undetermined cases**

1) $\lim\limits_{n\longrightarrow+\infty} U_n = +\infty \wedge \lim\limits_{n\longrightarrow+\infty} V_n = -\infty \Longrightarrow \lim\limits_{n\longrightarrow+\infty} (U_n + V_n) =?$

2) $\lim\limits_{n\longrightarrow+\infty} U_n = \pm\infty \wedge \lim\limits_{n\longrightarrow+\infty} V_n = 0 \Longrightarrow \lim\limits_{n\longrightarrow+\infty} (U_n.V_n) =?$

## 2.5 Subsequences (or extracted sequences)

**Definition 77** *Let $(U_n)_n$ be a numerical sequence and $(n_k)_k$ a strictly increasing sequence of integers. $(U_{n_k})$ is said to be a subsequence of $(U_n)_n$.*

**Example 78** *Let the sequence of general term $U_n = \dfrac{1}{n}, \quad n \in \mathbb{N}$.*

$U_{2n} = \dfrac{1}{2n},$

$U_{2n+1} = \dfrac{1}{2n+1},$

$U_{3n} = \dfrac{1}{3n},$

*$(U_{2n}), (U_{2n+1})$ and $(U_{3n})$ are subsequences of $(U_n)$.*

**Theorem 79** *If $(U_n)$ is a sequence converging to $\ell$, then every subsequence of $(U_n)$ converges to the same limit $\ell$.*

**Remark 80** *The converse of this theorem is not true, meaning that one can extract a convergent subsequence from a divergent sequence.*

**Example 81** *$U_n = (-1)^n$ diverges while its two subsequences ($U_{2n} = 1$) and ($U_{2n+1} = -1$) converge.*

**Theorem 82** *The sequence $(U_n)_n$ converges to $\ell \Longleftrightarrow$ the subsequences ($U_{2n}$) and ($U_{2n+1}$) converge to the same limit $\ell$.*

**Remark 83** *If $\lim\limits_{n \longrightarrow +\infty} U_{2n} \neq \lim\limits_{n \longrightarrow +\infty} U_{2n+1}$, then the sequence $(U_n)_n$ diverges.*

**Example 84** *Let $U_n = (-1)^n$.*

$\lim\limits_{n \longrightarrow +\infty} U_{2n} = 1 \neq \lim\limits_{n \longrightarrow +\infty} U_{2n+1} = -1$, *then $(U_n)_n$ diverges.*

**Theorem 85** ***(Bolzano-Weierstrass theorem)***

*Every bounded numerical sequence has a convergent subsequence.*

**Example 86** *The sequence with general term $U_n = (-1)^n$ diverges but it is bounded. Its two subsequences ($U_{2n} = 1$) and ($U_{2n+1} = -1$) converge.*

## 2.6 Adjacent sequences

**Definition 87** *Let $(U_n)_n$ and $(V_n)_n$ be two numerical sequences. We say that $(U_n)_n$ and $(V_n)_n$ are two adjacent sequences if one is increasing and the other decreasing and $\lim\limits_{n \longrightarrow +\infty} (U_n - V_n) = 0$.*

**Theorem 88** *If $(U_n)_n$ and $(V_n)_n$ are two adjacent numerical sequences, then they converge to the same limit.*

## 2.7 The Cauchy sequences

**Definition 89** *The numerical sequence $(U_n)_n$ is said to be a Cauchy sequence if*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p, q \in \mathbb{N}, (p \geq N \wedge q \geq N \Longrightarrow |U_p - U_q| < \varepsilon),$$

*or*

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, p \in \mathbb{N}, (n \geq N \Longrightarrow |U_{n+p} - U_n| < \varepsilon).$$

**Theorem 90** *Let $(U_n)_n$ be a convergente sequence, then $(U_n)_n$ is a Cauchy sequence.*

**Theorem 91** ***(Cauchy's Criterion)***

*Let $(U_n)_n$ be a Cauchy numerical sequence, then $(U_n)_n$ is a convergente sequence. We say that $\mathbb{R}$ is a complete space.*

**Remark 92** *To show that a numerical sequence is convergent, it suffices to show that it is a Cauchy sequence.*

## 2.8 Exercises

**Exercise 93** *Let $(U_n)$ be a sequence defined by*

$$\begin{cases} U_1 & = & 1 \ , \ U_2 = 2, \\ U_n & = & \dfrac{U_{n-1} + 2U_{n-2}}{3} \ , \ n \geq 3. \end{cases}$$

*We set :* $\forall n \geq 2 \ , \quad V_n = U_n - U_{n-1}$.

1) *Prove by induction that :* $\forall n \geq 2 \ , \ V_n = \left(\dfrac{-2}{3}\right)^{n-2}$.

2) *Prove that the sequence defined by :* $\forall n \geq 2 \ , \ S_n = V_2 + V_3 + ..... + V_n$ *is convergent and calculate its limit.*

3) *Deduce that the sequence $(U_n)$ is convergent and calculate its limit.*

**Solution** :
Let $(U_n)$ be the sequence defined by

$$\begin{cases} U_1 & = & 1 \ , \ U_2 = 2, \\ U_n & = & \dfrac{U_{n-1} + 2U_{n-2}}{3} \ , \ n \geq 3. \end{cases}$$

We set : $\forall n \geq 2 \ , \quad V_n = U_n - U_{n-1}$.

1) We prove by induction that : $\forall n \geq 2 \ , \ V_n = \left(\dfrac{-2}{3}\right)^{n-2}$.

For $n = 2, V_2 = U_2 - U_1 = 2 - 1 = 1 = \left(\dfrac{-2}{3}\right)^{2-2}$ : it is verified.

We assume that the property is true up to order $n$ and we show that it is true for order $(n + 1)$.

We assume that $V_n = \left(\dfrac{-2}{3}\right)^{n-2}$ and we show that $V_{n+1} = \left(\dfrac{-2}{3}\right)^{n-1}$.

$V_{n+1} = U_{n+1} - U_n = \dfrac{U_n + 2U_{n-1}}{3} - U_n = -\dfrac{2}{3}(U_n - U_{n-1})$,

$V_{n+1} = -\dfrac{2}{3}V_n = -\dfrac{2}{3}\left(\dfrac{-2}{3}\right)^{n-2} = \left(\dfrac{-2}{3}\right)^{n-1}$.

Thus, $\forall n \geq 2 \ , \ V_n = \left(\dfrac{-2}{3}\right)^{n-2}$.

2) We prove that the sequence defined by : $\forall n \geq 2 \ , \ S_n = V_2 + V_3 + ..... + V_n$ is convergent :

$S_n = V_2 + V_3 + ..... + V_n = 1 + \left(\frac{-2}{3}\right) + \left(\frac{-2}{3}\right)^2 + ..... + \left(\frac{-2}{3}\right)^{n-2},$

It is the sum of $(n-1)$ consecutive terms of a geometric sequence with ratio $q = \frac{-2}{3}$,

therefore, $S_n = \dfrac{1-\left(\frac{-2}{3}\right)^{n-1}}{1-\left(\frac{-2}{3}\right)} = \frac{3}{5}\left(1-\left(\frac{-2}{3}\right)^{n-1}\right).$

Since $\left|\frac{-2}{3}\right| < 1$, then $\lim\limits_{n\longrightarrow+\infty}\left(\frac{-2}{3}\right)^{n-1} = 0,$

hence, $\lim\limits_{n\longrightarrow+\infty} S_n = \lim\limits_{n\longrightarrow+\infty}\frac{3}{5}\left(1-\left(\frac{-2}{3}\right)^{n-1}\right) = \frac{3}{5}.$

3) We deduce that the sequence $(U_n)$ is convergent :

$S_n = V_2 + V_3 + .... + V_n = (U_2 - U_1) + (U_3 - U_2) + .... + (U_n - U_{n-1}),$

$S_n = U_n - U_1 = U_n - 1$, thus $U_n = S_n + 1$.

Since $(S_n)$ converge, then $(U_n)$ converge and we have

$\lim\limits_{n\longrightarrow+\infty} U_n = \lim\limits_{n\longrightarrow+\infty}(S_n + 1) = \frac{3}{5} + 1 = \frac{8}{5}.$

**Exercise 94** *Let $(U_n)$ be a sequence defined by*

$$\begin{cases} U_0 & = & 1, \\ U_{n+1} & = & \dfrac{1+U_n}{3+U_n} \end{cases}, \quad n \in \mathbb{N}.$$

1) *Prove that $\forall n \in \mathbb{N}\ ,\ \ 0 \leq U_n \leq 1$.*

2) *Prove that $(U_n)$ is monotonic.*

3) *Deduce that $(U_n)$ is convergent and calculate its limit $\ell$.*

4) *Let $E = \{U_n\ /\ n \in \mathbb{N}\}$. Determine* $\sup E$ *and* $\inf E$.

**Solution** :
Let $(U_n)$ be the sequence defined by

$$\begin{cases} U_0 & = & 1, \\ U_{n+1} & = & \dfrac{1+U_n}{3+U_n} \end{cases}, \quad n \in \mathbb{N}.$$

1) Prove by induction that $\forall n \in \mathbb{N}\ ,\ \ 0 \leq U_n \leq 1,$

for $n = 0,\ \ 0 \le U_0 = 1 \le 1$ : it is verified.

We assume that the property is true up to order $n$ and we show that it is true for order $(n+1)$.

We assume that $0 \le U_n \le 1$ and we show that $0 \le U_{n+1} \le 1$.

$U_n \ge 0 \Longrightarrow U_{n+1} = \dfrac{1+U_n}{3+U_n} \ge 0.$

$U_{n+1} - 1 = \dfrac{1+U_n}{3+U_n} - 1 = \dfrac{-2}{U_n+3} \le 0$

then, $0 \le U_{n+1} \le 1$.

Therefore, $\forall n \in \mathbb{N}, \qquad 0 \le U_n \le 1.$

2) We prove that $(U_n)$ is monotonic :

$U_{n+1} = \dfrac{1+U_n}{3+U_n} = f(U_n).$

We set $f(x) = \dfrac{x+1}{x+3},\ \ x \in D = [0, +\infty[\,.$

We have $f(D) \subset D$ and $f'(x) = \dfrac{2}{(x+3)^2} > 0,$

then, $f$ is increasing, hence $(U_n)_n$ is monotonic.

Since $U_1 = \dfrac{U_0+1}{U_0+3} = \dfrac{1}{2} < U_o = 1$, thus $(U_n)_n$ is decreasing.

3) We deduce that $(U_n)$ is convergent :

$(U_n)_n$ is decreasing and bounded below, then it converges to its infimum, i.e. $\lim\limits_{n \longrightarrow +\infty} U_n = \ell = \inf(U_n).$

We calculate the limit of $(U_n)$ :

$\ell = \lim\limits_{n \longrightarrow +\infty} U_n \Longrightarrow \ell = f(\ell) \Longrightarrow \ell = \dfrac{\ell+1}{\ell+3} \Longleftrightarrow \ell^2 + 2\ell - 1 = 0,$

$\Delta = 8,\ \ \ell_1 = -1 + \sqrt{2} > 0\ \ \text{and}\ \ \ell_2 = -1 - \sqrt{2} < 0,$

since $\ 0 \le U_n \le 1$, then $\ \ell = -1 + \sqrt{2}$.

4) Let $E = \{U_n\ /\ n \in \mathbb{N}\}$ , we determine $\sup E$ and $\inf E$ :

$\sup E = U_0 = 1$, (since $(U_n)_n$ is decreasing, $\forall n \in \mathbb{N}, U_0 \ge U_n$ and $U_0 \in E$).

$\inf E = \ell = -1 + \sqrt{2}$.

**Exercise 95** *Using adjacent sequences, show that the following sequence is convergent :*

$$U_n = 1 - \frac{1}{2} + \frac{1}{3} - .......... + \frac{(-1)^{n-1}}{n} \quad , \quad n \in \mathbb{N}^*.$$

**Solution** :

We show that $(U_{2n})$ and $(U_{2n+1})$ are adjacent :

i.e. one is increasing, the other is decreasing, and $\lim\limits_{n \longrightarrow +\infty} (U_{2n+1} - U_{2n}) = 0.$

• We study the monotonicity of $(U_{2n})$ :

$U_{2n+2} - U_{2n} = \left(1 + .. + \frac{(-1)^{2n-1}}{2n} + \frac{(-1)^{2n}}{2n+1} + \frac{(-1)^{2n+1}}{2n+2}\right) - \left(1 + .. + \frac{(-1)^{2n-1}}{2n}\right),$

$U_{2n+2} - U_{2n} = \frac{1}{2n+1} - \frac{1}{2n+2} = \frac{1}{(2n+1)(2n+2)} > 0,$

then $(U_{2n})$ is increasing.

• We study the monotonicity of $(U_{2n+1})$ :

$U_{2n+3} - U_{2n+1} = \left(1 + .. + \frac{(-1)^{2n}}{2n+1} + \frac{(-1)^{2n+1}}{2n+2} + \frac{(-1)^{2n+2}}{2n+3}\right) - \left(1 + .. + \frac{(-1)^{2n}}{2n+1}\right),$

$U_{2n+2} - U_{2n} = -\frac{1}{2n+2} + \frac{1}{2n+3} = \frac{-1}{(2n+2)(2n+3)} < 0,$

then, $(U_{2n+1})$ is decreasing.

• $\lim\limits_{n \longrightarrow +\infty} (U_{2n+1} - U_{2n}) = \lim\limits_{n \longrightarrow +\infty} \frac{1}{2n+1} = 0.$

hence, $(U_{2n})$ and $(U_{2n+1})$ are adjacent,

therefore, $(U_{2n})$ and $(U_{2n+1})$ converge to the same limit and so $(U_n)$ is convergent.

**Exercise 96** 1) *Let $(U_n)$ be the sequence defined by : $U_n = \sum\limits_{k=1}^{n} \frac{\cos k}{2^k} \quad , \quad n \in \mathbb{N}^*$.*

*Prove that $(U_n)$ is a Cauchy sequence. What can we deduce?*

2) *Let $(U_n)$ the sequence defined by : $U_n = 1 + \frac{1}{2} + \frac{1}{3} + .......... + \frac{1}{n}, \quad n \in \mathbb{N}^*$.*

*Prove that $(U_n)$ is not a Cauchy sequence and deduce its limit.*

**Solution** :

1) Let $(U_n)$ be the sequence defined by : $U_n = \sum\limits_{k=1}^{n} \frac{\cos k}{2^k} \quad , \quad n \in \mathbb{N}^*.$

We prove that $(U_n)$ is a Cauchy sequence :

$((U_n)$ is a Cauchy sequence$) \Longleftrightarrow$

$(\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, p \in \mathbb{N}, (n \geq N \Longrightarrow |U_{n+p} - U_n| < \varepsilon).$

Let $\varepsilon > 0$,

$|U_{n+p} - U_n| = \left| \sum\limits_{k=1}^{n+p} \dfrac{\cos k}{2^k} - \sum\limits_{k=1}^{n} \dfrac{\cos k}{2^k} \right|$,

$|U_{n+p} - U_n| = \left| \dfrac{\cos(n+1)}{2^{n+1}} + \dfrac{\cos(n+2)}{2^{n+2}} + ..... + \dfrac{\cos(n+p)}{2^{n+p}} \right|$,

$|U_{n+p} - U_n| \leq \left| \dfrac{\cos(n+1)}{2^{n+1}} \right| + \left| \dfrac{\cos(n+2)}{2^{n+2}} \right| + ..... + \left| \dfrac{\cos(n+p)}{2^{n+p}} \right|$,

$|U_{n+p} - U_n| \leq \dfrac{1}{2^{n+1}} + \dfrac{1}{2^{n+2}} + ..... + \dfrac{1}{2^{n+p}} = \dfrac{1}{2^{n+1}} \left( 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + .... + \dfrac{1}{2^{p-1}} \right)$,

$|U_{n+p} - U_n| \leq \dfrac{1}{2^{n+1}} \left( 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + .... + \dfrac{1}{2^{p-1}} \right) = \dfrac{1}{2^{n+1}} \left( \dfrac{1 - \left(\frac{1}{2}\right)^p}{1 - \frac{1}{2}} \right)$,

$|U_{n+p} - U_n| \leq \dfrac{1}{2^n} \left( 1 - \left(\frac{1}{2}\right)^p \right) < \dfrac{1}{2^n}$,

so it is enough to take $\dfrac{1}{2^n} < \varepsilon$.

$\dfrac{1}{2^n} < \varepsilon \Longleftrightarrow 2^n > \dfrac{1}{\varepsilon} \Longleftrightarrow n \ln 2 > \ln \dfrac{1}{\varepsilon} \Longleftrightarrow n > \frac{\ln \frac{1}{\varepsilon}}{\ln 2}$,

$n > \frac{\ln \frac{1}{\varepsilon}}{\ln 2} \Longrightarrow n \geq \left[ \left| \frac{\ln \frac{1}{\varepsilon}}{\ln 2} \right| \right] + 1$,

so it is enough to take $N = \left[ \left| \frac{\ln \frac{1}{\varepsilon}}{\ln 2} \right| \right] + 1$.

Then $(U_n)$ is a Cauchy sequence and since $\mathbb{R}$ is a complete space, then $(U_n)$ is a convergent sequence.

2) Let $(U_n)$ the sequence defined by : $U_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + .... + \dfrac{1}{n}$ , $n \in \mathbb{N}^*$.

We prove that $(U_n)$ is not a Cauchy sequence :

($(U_n)$ is not a Cauchy sequence) $\Longleftrightarrow$

$(\exists \varepsilon > 0, \forall N \in \mathbb{N}^*, \exists p, q \in \mathbb{N}^*, (p \geq N \wedge q \geq N$ and $|U_p - U_q| \geq \varepsilon)$.

Let $N \in \mathbb{N}^*$, we set $p = 2N \geq N$ and $q = N \geq N$,

$|U_p - U_q| = |U_{2N} - U_N| \left| \left( 1 + \dfrac{1}{2} + .. + \dfrac{1}{N} + .. + \dfrac{1}{2N} \right) - \left( 1 + \dfrac{1}{2} + ... + \dfrac{1}{N} \right) \right|$,

$|U_p - U_q| = \dfrac{1}{N+1} + \dfrac{1}{N+2} + ... + \dfrac{1}{2N} \geq \dfrac{1}{2N} + \dfrac{1}{2N} + ... + \dfrac{1}{2N} = N \dfrac{1}{2N} = \dfrac{1}{2}$,

$|U_p - U_q| \geq \dfrac{1}{2} = \varepsilon$, so, it is enough to take $\varepsilon = \dfrac{1}{2}$.

therefore $(U_n)$ is not a Cauchy sequence, which implies that $(U_n)$ diverge. Since $(U_n)$ is increasing, so it is not bounded (otherwise it would be convergent), thus, $\lim\limits_{n \longrightarrow +\infty} U_n = +\infty$.

# Chapter 3

# Functions of one real variable. Limit and continuity

## 3.1 Generalities

### 3.1.1 Definition of a function

**Definition 97** *A numerical function on a set $X$ is any mapping from $X$ to the set.$\mathbb{R}$ of real numbers. If $X \subseteq \mathbb{R}$, we say that $f$ is a numerical function of a real variable.*

$$\begin{array}{rccc} f: & X \subseteq \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longrightarrow & f(x). \end{array}$$

### 3.1.2 Domain of definition

Let $f$ be a real function of a real variable. The domain of definition of the function $f$ is the set defined by

$$D_f = \{x \in \mathbb{R} \ / \ f(x) \text{ is defined}\}.$$

**Example 98** 1) $f(x) = e^{\frac{1}{x^2-1}}$,

$$D_f = \left\{x \in \mathbb{R}/x^2 - 1 \neq 0\right\} = \{x \in \mathbb{R}/x \neq \pm 1\} = \mathbb{R} \backslash \{-1, 1\}.$$

2) $f(x) = \dfrac{\ln(x+1)}{x}$,

$$D_f = \{x \in \mathbb{R}/x + 1 > 0 \wedge x \neq 0\} = \{x \in \mathbb{R}/x > -1 \wedge x \neq 0\} = ]-1, 0[\cup]0, +\infty[.$$

### 3.1.3 Graph of a real function

Let $f : X \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, be a numerical function.defined on $X$.

The graph of the function $f$ is a part of $\mathbb{R}^2$ defined by the following set

$$Gr(f) = \left\{(x, f(x)) \in \mathbb{R}^2 \ / \ x \in X\right\} \subset \mathbb{R}^2.$$

**Example 99** $f : \mathbb{R} \longrightarrow \mathbb{R} \ / \ f(x) = x^2$

$Gr(f) = \left\{(x, x^2) \in \mathbb{R}^2 \ / \ x \in \mathbb{R}\right\}$ : *It is a parabola.*

### 3.1.4 Algebraic operations on functions

- Let $f$ and $g$ be two real functions defined on $X \subset \mathbb{R}$.

$(f = g) \iff (\forall x \in X, \ f(x) = g(x))$.

- We note $F(X, \mathbb{R})$ the set of functions from $X$ to $\mathbb{R}$,

$F(X, \mathbb{R}) = \{f : X \longrightarrow \mathbb{R}\}$.

- We define two internal operations, addition and multiplication on $F(X, \mathbb{R})$ by

$\forall x \in X, (f + g)(x) = f(x) + g(x),$

$$(f.g)(x) = f(x).g(x).$$

$F(X, \mathbb{R})$ equipped with these two internal operations is a unitary commutative ring.

- We define an external operation on $F(X, \mathbb{R})$ by

$\forall \lambda \in \mathbb{R}, \forall x \in X, (\lambda f)(x) = \lambda.f(x)$.

- $F(X, \mathbb{R})$ equipped with addition and this external operation is a vector space on $\mathbb{R}$.

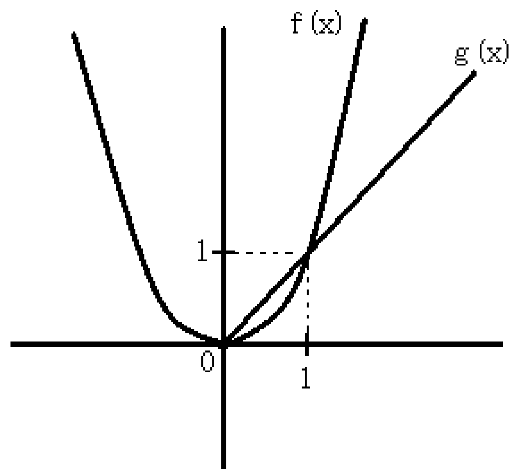
- We define the order relation on $F(X, \mathbb{R})$ by

$(f \leq g) \iff (\forall x \in X, \ f(x) \leq g(x))$.

This relation is not a relation of total order.

Indeed, we take $f : \mathbb{R} \longrightarrow \mathbb{R} \ / \ f(x) = x^2$ and $g : \mathbb{R} \longrightarrow \mathbb{R}/g(x) = x$.

We notice that $f \nleq g$ and $g \nleq f$.




### 3.1.5 Even, odd and periodic functions

Let $f$ be a function defined on a symmetric interval $I$ with respect to 0.

- $f$ is even $\Longleftrightarrow \forall x \in I,\ f(-x) = f(x)$,
- $f$ is odd $\Longleftrightarrow \forall x \in I,\ f(-x) = -f(x)$.
- A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be periodic if

$$\exists P > 0, \forall x \in \mathbb{R}, f(x+P) = f(x).$$

We say that $P$ is the period of $f$.

If $P$ is a period of $f$, then every number of the form $kP$ ($k \in \mathbb{N}^*$) is a period of $f$ i.e., $\forall x \in \mathbb{R},\ f(x+kP) = f(x)$.

**Example 100** 1) $f(x) = \cos x,\ x \in \mathbb{R}$ *is an even and periodic function of period* $P = 2\pi$.

$\forall x \in \mathbb{R},\ \cos(-x) = \cos(x)$ *and* $\cos(x+2\pi) = \cos(x)$.

2) $f(x) = \sin x,\ x \in \mathbb{R}$ *is an odd and periodic function of period* $P = 2\pi$.

$\forall x \in \mathbb{R},\ \sin(-x) = -\sin(x)$ *and* $\sin(x+2\pi) = \sin(x)$.

**Remark 101** 1) *The graph of an even function has an axis of symmetry (the y-axis).*

2) *The graph of an odd function has a center of symmetry (the origin (O)).*

### 3.1.6 Bounded functions and monotonic functions

Let $f : X \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function.

- $f$ is bounded above $\Longleftrightarrow (\exists M \in \mathbb{R}, \forall x \in X,\ f(x) \leq M)$.
- $f$ is bounded below $\Longleftrightarrow (\exists m \in \mathbb{R}, \forall x \in X,\ f(x) \geq m)$.
- $f$ is bounded $\Longleftrightarrow$ $f$ is bounded above and bounded below.
- $f$ is bounded $\Longleftrightarrow (\exists M, m \in \mathbb{R}, \forall x \in X,\ m \leq f(x) \leq M)$.
- $f$ is bounded $\Longleftrightarrow (\exists \alpha > 0, \forall x \in X,\ |f(x)| \leq \alpha)$.
- $f$ is increasing $\Longleftrightarrow (\forall x, y \in X,\ x \leq y \Longrightarrow f(x) \leq f(y))$.
- $f$ is strictly increasing $\Longleftrightarrow (\forall x, y \in X,\ x < y \Longrightarrow f(x) < f(y))$.
- $f$ is decreasing $\Longleftrightarrow (\forall x, y \in X,\ x \leq y \Longrightarrow f(x) \geq f(y))$.
- $f$ is strictly decreasing $\Longleftrightarrow (\forall x, y \in X,\ x < y \Longrightarrow f(x) > f(y))$.

**Example 102** 1) $f(x) = \cos x,\ \forall x \in \mathbb{R},\ |\cos x| \leq 1,$

$g(x) = \sin x,\ \forall x \in \mathbb{R},\ |\sin x| \leq 1,$

$\cos x$ *and* $\sin x$ *are bounded functions on* $\mathbb{R}$.

2) $f(x) = x^2$, *it is an even function but it is not bounded on* $\mathbb{R}$.

3) $f(x) = \dfrac{1}{x}$, *it is an odd, unbounded function on* $\mathbb{R}^*$.

## 3.2 Limit of a function at point $x_0$

**Definition 103** *Let $x_0 \in \mathbb{R}$. A part $V \subset \mathbb{R}$ is a neighborhood of $x_0$ if it contains an open interval containing $x_0$.*

**Definition 104** *We say that a function $f$, defined in the neighborhood of $x_0$ except maybe in $x_0$, has a limit $\ell \in \mathbb{R}$ at point $x_0$ if*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in D_f, x \neq x_0, (|x - x_0| < \alpha \Longrightarrow |f(x) - \ell| < \varepsilon)$$

*We write* $\lim\limits_{x \longrightarrow x_0} f(x) = \ell$

Hence

$$\lim_{x\longrightarrow x_0} f(x)=\ell \Leftrightarrow (\forall\varepsilon>0,\exists\alpha>0,\forall x\in X, x\neq x_0,(|x-x_0|<\alpha \Longrightarrow |f(x)-\ell|<\varepsilon))$$

**Example 105** 1) $\lim_{x\longrightarrow 1}(2x+1)=3$

$\Leftrightarrow (\forall\varepsilon>0,\exists\alpha>0,\forall x\in\mathbb{R}, x\neq 1,(|x-1|<\alpha \Longrightarrow |2x+1-3|<\varepsilon))$.

*Let* $\varepsilon>0$,

$|2x+1-3|<\varepsilon \Longleftrightarrow |2x-2|<\varepsilon \Longleftrightarrow 2|x-1|<\varepsilon \Longleftrightarrow |x-1|<\dfrac{\varepsilon}{2}=\alpha$,

*then it suffices to take* $\alpha=\dfrac{\varepsilon}{2}$.

2) $\lim_{x\longrightarrow 0}x^2=0 \Leftrightarrow \left(\forall\varepsilon>0,\exists\alpha>0,\forall x\in\mathbb{R}, x\neq 0,(|x-0|<\alpha \Longrightarrow \left|x^2-0\right|<\varepsilon)\right)$.

*Let* $\varepsilon>0$,

$\left|x^2\right|<\varepsilon \Longleftrightarrow x^2<\varepsilon \Longleftrightarrow \sqrt{x^2}<\sqrt{\varepsilon} \Longleftrightarrow |x|<\sqrt{\varepsilon}=\alpha$,

*then it suffices to take* $\alpha=\sqrt{\varepsilon}$.

### 3.2.1 The limit of $f$ to the right and to the left of $x_0$

Let $f$ be a numerical function.

**The limit of $f$ to the right of $x_0$**: Let $f$ be a function defined to the right of $x_0$,

$$\lim_{x\overset{>}{\longrightarrow}x_0} f(x)=\ell \Leftrightarrow (\forall\varepsilon>0,\exists\alpha>0,\forall x\in D_f,(0<x-x_0<\alpha \Longrightarrow |f(x)-\ell|<\varepsilon)).$$

**The limit of $f$ to the left of $x_0$** : Let $f$ be a function defined to the left of $x_0$,

$$\lim_{x\overset{<}{\longrightarrow}x_0} f(x)=\ell \Leftrightarrow (\forall\varepsilon>0,\exists\alpha>0,\forall x\in D_f,(-\alpha<x-x_0<0 \Longrightarrow |f(x)-\ell|<\varepsilon)).$$

**Conclusion:** Let $f$ be a function defined to the right and to the left of $x_0$,

$$\lim_{x\longrightarrow x_0} f(x)=\ell \Leftrightarrow \left(\lim_{x\overset{>}{\longrightarrow}x_0} f(x)=\lim_{x\overset{<}{\longrightarrow}x_0} f(x)=\ell\right).$$

**Example 106** *Let* $f(x)=\dfrac{|x|}{x}$, $x_0=0$, $D_f=\mathbb{R}^*$.

$\lim_{x\overset{>}{\longrightarrow}0} f(x)=\lim_{x\overset{>}{\longrightarrow}0}=\dfrac{x}{x}=1$,

$\lim_{x\overset{<}{\longrightarrow}0} f(x)=\lim_{x\overset{<}{\longrightarrow}0}=\dfrac{-x}{x}=-1$,

*then* $\lim_{x\overset{>}{\longrightarrow}0} f(x)\neq\lim_{x\overset{<}{\longrightarrow}0} f(x)$ *and so* $\lim_{x\longrightarrow x_0} f(x)$ *does not exist.*

### 3.2.2 Extension of the notion of limit

**Case where $x_0$ is infinite**

1) $\lim\limits_{x \longrightarrow +\infty} f(x) = \ell \Longleftrightarrow (\forall \varepsilon > 0, \exists B > 0, \forall x \in X, (x > B \Longrightarrow |f(x) - \ell| < \varepsilon))$.

2) $\lim\limits_{x \longrightarrow -\infty} f(x) = \ell \Longleftrightarrow (\forall \varepsilon > 0, \exists B > 0, \forall x \in X, (x < -B \Longrightarrow |f(x) - \ell| < \varepsilon))$.

3) $\lim\limits_{x \longrightarrow +\infty} f(x) = +\infty \Longleftrightarrow (\forall A > 0, \exists B > 0, \forall x \in X, (x > B \Longrightarrow f(x) > A))$.

4) $\lim\limits_{x \longrightarrow +\infty} f(x) = -\infty \Longleftrightarrow (\forall A > 0, \exists B > 0, \forall x \in X, (x > B \Longrightarrow f(x) < -A))$.

5) $\lim\limits_{x \longrightarrow -\infty} f(x) = +\infty \Longleftrightarrow (\forall A > 0, \exists B > 0, \forall x \in X, (x < -B \Longrightarrow f(x) > A))$.

6) $\lim\limits_{x \longrightarrow -\infty} f(x) = -\infty \Longleftrightarrow (\forall A > 0, \exists B > 0, \forall x \in X, (x < -B \Longrightarrow f(x) < -A))$.

**Case where $x_0$ is finite and the limit infinite**

1) $\lim\limits_{x \longrightarrow x_0} f(x) = +\infty$

$$\Longleftrightarrow (\forall A > 0, \exists \alpha > 0, \forall x \in X, x \neq x_0, (|x - x_0| < \alpha \Longrightarrow f(x) > A)).$$

2) $\lim\limits_{x \longrightarrow x_0} f(x) = -\infty$

$$\Longleftrightarrow (\forall A > 0, \exists \alpha > 0, \forall x \in X, x \neq x_0, (|x - x_0| < \alpha \Longrightarrow f(x) < -A)).$$

## 3.3 Main theorems on limits

**Theorem 107** ***(uniqueness of the limit)***
*If $f$ admits a limit $\ell$ at the point $x_0$, so this limit is unique.*

**Theorem 108** *Let $f : X \longrightarrow \mathbb{R}$, $x_0$ an adherent point of $X$ and $\ell \in \mathbb{R}$ (or $\ell = \pm\infty$) then, we have*

$$\lim_{x \longrightarrow x_0} f(x) = \ell \Longleftrightarrow \left( \forall (x_n) \subset X, \left( \lim_{n \longrightarrow +\infty} x_n = x_0 \Longrightarrow \lim_{n \longrightarrow +\infty} f(x_n) = \ell \right) \right).$$

**Remark 109** *We use this theorem to show that the limit of some functions does not exist.*

**Example 110** *Prove that* $\lim\limits_{x \longrightarrow 0} \sin \dfrac{1}{x}$ *does not exist.*

*We take the sequence* $x_n = \dfrac{1}{n\pi + \frac{\pi}{2}}$, $n \in \mathbb{N}$,

$$\lim_{n \longrightarrow +\infty} x_n = \lim_{n \longrightarrow +\infty} \frac{1}{n\pi + \frac{\pi}{2}} = 0,$$

$\lim_{n\longrightarrow+\infty} f(x_n) = \lim_{n\longrightarrow+\infty} \sin(n\pi + \frac{\pi}{2}) = \lim_{n\longrightarrow+\infty} (-1)^n$ : *does not exist.*

*Then* $\lim_{x\longrightarrow 0} \sin\frac{1}{x}$ *does not exist.*

*We show in the same way that* $\lim_{x\longrightarrow 0} \cos\frac{1}{x}$ *does not exist,*

*we take* $x_n = \frac{1}{n\pi}$, $n \in \mathbb{N}^*$.

**Theorem 111** *Let* $f, g : X \longrightarrow \mathbb{R}$ *and* $x_0$ *an adherent point of* $X$ *(or* $x_0 = \pm\infty$*). If* $g$ *is bounded and* $\lim_{x\longrightarrow x_0} f(x) = 0$, *then* $\lim_{x\longrightarrow x_0} (f.g)(x) = 0$.

**Example 112** $\lim_{x\longrightarrow 0} x^2 \sin\frac{1}{x} = 0$.

*Indeed,* $\sin\frac{1}{x}$ *is bounded and* $\lim_{x\longrightarrow 0} x^2 = 0$.

**Theorem 113** *Let* $f, g : X \longrightarrow \mathbb{R}$ *and* $x_0$ *an adherent point of* $X$ *(or* $x_0 = \pm\infty$*).*

*If* $\lim_{x\longrightarrow x_0} f(x) = \ell \in \mathbb{R}$ *and* $\lim_{x\longrightarrow x_0} g(x) = \ell' \in \mathbb{R}$, *then*

1) $\lim_{x\longrightarrow x_0} (f+g)(x) = \ell + \ell'$.

2) $\lim_{x\longrightarrow x_0} (f.g)(x) = \ell.\ell'$.

3) $\lim_{x\longrightarrow x_0} \lambda f(x) = \lambda\ell$.

4) $\lim_{x\longrightarrow x_0} \frac{1}{f(x)} = \frac{1}{\ell}$ $(\forall x \in X,\ f(x) \neq 0$ *and* $\ell \in \mathbb{R}^*)$.

5) $\lim_{x\longrightarrow x_0} f(x) = \pm\infty$ *and* $\forall x \in X,\ f(x) \neq 0 \Longrightarrow \lim_{x\longrightarrow x_0} \frac{1}{f(x)} = 0$.

**Remark 114** *If* $\forall x \in X,\ f(x) \leq g(x)$ *then* $\ell \leq \ell'$.

**Undetermined cases**

1) $+\infty - \infty =?$

2) $0.\infty =?$

3) $1^\infty =?$

**Example 115** 1) $\lim\limits_{x\longrightarrow a}\dfrac{x-a}{x^2-a^2}=\lim\limits_{x\longrightarrow a}\dfrac{x-a}{(x-a)(x+a)}=\lim\limits_{x\longrightarrow a}\dfrac{1}{x+a}=\dfrac{1}{2a}.$

2) $\lim\limits_{x\longrightarrow +\infty}\left(1+\dfrac{a}{x}\right)^x, \quad a\in\mathbb{R},$

$$\lim_{x\longrightarrow +\infty}\left(1+\frac{a}{x}\right)^x=\lim_{x\longrightarrow +\infty}e^{x\ln\left(1+\frac{a}{x}\right)}=e^{\lim\limits_{x\longrightarrow +\infty}x\ln\left(1+\frac{a}{x}\right)}=e^{\lim\limits_{x\longrightarrow +\infty}a\frac{\ln\left(1+\frac{a}{x}\right)}{\frac{a}{x}}}$$
$$=e^a.$$

**Remark 116** *We recall that*

1) $h(x)=(f(x))^{g(x)}=e^{g(x)\ln(f(x))}$,

$D_h=\{x\in\mathbb{R}/f(x)>0\}\cap D_g$.

2) $f(x)=\sqrt{x},\ \ D_f=[0,+\infty[$.

3) $g(x)=\sqrt[3]{x},\ \ D_g=\mathbb{R}$.

**Theorem 117** *Let $x_0$ an adherent point of $X$ and the functions $f,g,h:X\longrightarrow\mathbb{R}$ such that $\forall x\in X,\ g(x)\leq f(x)\leq h(x)$.*
*If $\lim\limits_{x\longrightarrow x_0}g(x)=\lim\limits_{x\longrightarrow x_0}h(x)=\ell$, then $\lim\limits_{x\longrightarrow x_0}f(x)=\ell$.*

## 3.4 Continuous functions

### 3.4.1 Definition of the continuity of a function

**Definition 118** *Let $f:X\longrightarrow\mathbb{R}$ and $x_0\in X=D_f$.*

*$f$ is continuous at the point $x_0 \Longleftrightarrow \lim\limits_{x\longrightarrow x_0}f(x)=f(x_0)$*

$\Leftrightarrow(\forall\varepsilon>0,\exists\alpha>0,\forall x\in X,x\neq x_0,(|x-x_0|<\alpha\Longrightarrow|f(x)-f(x_0)|<\varepsilon))$.

**Remark 119** 1) *We study the continuity of the function $f$ at the point $x_0$ which belongs to the domain of definition of $f$ i.e. $x_0\in D_f$.*

2) *$f$ is continuous on $X$ if it is continuous at every point $x_0\in X$.*

**Example 120** 1) *$f(x)=c$, the constant function is continuous on $\mathbb{R}$. Indeed,*

$\forall\varepsilon>0,\exists\alpha>0,\forall x\in\mathbb{R},x\neq x_0,(|x-x_0|<\alpha\Longrightarrow|f(x)-f(x_0)|=|c-c|=0<\varepsilon)$.

2) *$f(x)=x$, the identity function is continuous on $\mathbb{R}$. Indeed,*

$\forall\varepsilon>0,\exists\alpha>0,\forall x\in\mathbb{R},x\neq x_0,(|x-x_0|<\alpha\Longrightarrow|f(x)-f(x_0)|=|x-x_0|<\varepsilon=\alpha)$.

3) $f(x) = \sin x$, *is continuous on* $\mathbb{R}$. *Indeed,*

$\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in \mathbb{R}, x \neq x_0, (|x - x_0| < \alpha \Longrightarrow |\sin x - \sin x_0| < \varepsilon)$,

*We have* $|\sin x - \sin x_0| = \left|2\sin(\frac{x - x_0}{2})\cos(\frac{x + x_0}{2})\right| \leq 2\left|\sin(\frac{x - x_0}{2})\right| \leq$

$2\left|\frac{x - x_0}{2}\right| \leq |x - x_0| < \varepsilon = \alpha$, *since we have* $|\sin y| \leq |y|, \forall y \in \mathbb{R}$.

### 3.4.2 Right and left continuity of the function at $x_0$

**Right-hand continuity of the function $f$ at $x_0$**

$f$ is right-continuous at $x_0 \Longleftrightarrow \lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = f(x_0)$

$\Leftrightarrow (\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in X, (0 < x - x_0 < \alpha \Longrightarrow |f(x) - f(x_0)| < \varepsilon))$.

**Leftt-hand continuity of the function $f$ at $x_0$**

$f$ is left-continuous at $x_0 \Longleftrightarrow \lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$

$\Leftrightarrow (\forall \varepsilon > 0, \exists \alpha > 0, \forall x \in X, (-\alpha < x - x_0 < 0 \Longrightarrow |f(x) - f(x_0)| < \varepsilon))$.

**Conclusion**

$f$ is continuous at point $x_0 \Longleftrightarrow f$ is continuous from the right and from the left at $x_0$,

$$\lim_{x \longrightarrow x_0} f(x) = f(x_0) \Leftrightarrow \left(\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)\right).$$

### 3.4.3 Theorems and properties of continuous functions

**Theorem 121** *Let* $f : X \longrightarrow \mathbb{R}$ *and* $x_0 \in X = D_f$.

$f$ *is continuous at point* $x_0 \Longleftrightarrow$

$$\left(\forall (x_n) \subset X, \left(\lim_{n \longrightarrow +\infty} x_n = x_0 \Longrightarrow \lim_{n \longrightarrow +\infty} f(x_n) = f(x_0)\right)\right).$$

**Theorem 122** *Let* $f, g : X \longrightarrow \mathbb{R}$ *and* $x_0 \in X = D_f = D_g$.

*If* $f$ *and* $g$ *are continuous at point* $x_0$, *then*

1) $(f + g)$ *is continuous at point* $x_0$.

2) $(f.g)$ *is continuous at point* $x_0$.

3) $\lambda f \quad (\lambda \in \mathbb{R})$ *is continuous at point* $x_0$.

4) $|f|$ *is continuous at point* $x_0$.

5) $\dfrac{f}{g}$ *is continuous at point* $x_0$ *(if* $g(x) \neq 0$*).*

**Remark 123** *The function $f(x) = x$ is continuous on $\mathbb{R}$,*

*then $g(x) = x^n$ $(n \in \mathbb{N})$ is continuous on $\mathbb{R}$,*

*therefore $h(x) = \sum_{i=0}^{n} a_i x^i$ is continuous on $\mathbb{R}$.*

**Theorem 124** *Let $f : X \longrightarrow \mathbb{R}, g : Y \longrightarrow \mathbb{R} / f(X) \subset Y$ and $x_0 \in X$. We define the function $g \circ f : X \longrightarrow \mathbb{R}$ by*
$\forall x \in X$, $(g \circ f)(x) = g(f(x))$.

*If $f$ is continuous at $x_0$ and $g$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at $x_0$.*

**Example 125** *Let the function defined by*

$$f(x) = \begin{cases} \dfrac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

*Study the continuity of $f$ on $D_f$.*

$D_f = \mathbb{R}$

***Continuity of $f$ on*** $\mathbb{R}^*$ :

*$f$ is continuous on $\mathbb{R}^*$ because it is the composition and the product of continuous functions on $\mathbb{R}^*$.*

***Continuity of $f$ at point*** $x_0 = 0 : f(0) = 1$,

$$\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} \frac{\sin x}{x} = 1 = f(0),$$

$$\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} \frac{-\sin x}{x} = -1 \neq f(0).$$

***Conclusion :***
*$f$ is not continuous at* 0.

### 3.4.4 Main properties of continuous functions on an interval

**Theorem 126** *Any continuous numerical function on a bounded closed interval $[a, b]$ is bounded.*

**Remark 127** *This theorem is not true if the interval is not closed or is not bounded.*

**Example 128** *The function* $f(x) = \dfrac{1}{x}$ *is continuous on* $]0,1]$*, but it is not bounded on* $]0,1]$.

**Theorem 129** *Any continuous numerical function on a bounded closed interval* $[a,b]$, *reaches its supremum and infimum at least once,*

*i.e. there exist* $x_1, x_2 \in [a,b]$ *such that* $f(x_1) = \sup\limits_{x\in[a,b]} f(x)$ *and* $f(x_2) = \inf\limits_{x\in[a,b]} f(x)$.

**Remark 130** *This theorem is not true if the interval is not closed or is not bounded.*

**Example 131** *The function* $f(x) = x^2$ *is continuous on* $]-1,1[$,

*it is bounded :* $\forall x \in\, ]-1,1[\,, 0 \leq f(x) < 1$,

*it reaches its infimum :* $\inf\limits_{x\in]-1,1[} f(x) = 0 = f(0)$ *et* $0 \in\, ]-1,1[$,

*but it does not reach its supremum :*

$\sup\limits_{x\in]-1,1[} f(x) = 1 = f(1) = f(-1)$ *but* $1, -1 \notin\, ]-1,1[$.

**Theorem 132** ***(Intermediate Value Theorem)***
*Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a continuous function such that* $f(a).f(b) < 0$*, then there exists* $c \in\, ]a,b[\ /\ f(c) = 0$.

**Example 133** 1) *Prove that the equation* $x - 2 + \ln x = 0$ *admits a solution in* $]1, \sqrt{e}[$.

2) *Prove that this solution is unique.*

**Solution**
1) We set $f(x) = x - 2 + \ln x$, $D_f = ]0, +\infty[$.

$f$ is continuous on $]0,+\infty[$ because it is the sum of continuous functions on $]0,+\infty[$, especially $f$ is continuous on $[1,\sqrt{e}]$. On the other hand, we have $f(1).f(\sqrt{e}) < 0$. So according to the intermediate value theorem, there exists

$c \in\, ]1, \sqrt{e}[\ /\ f(c) = 0$.

2) To prove that the solution $c$ is unique, it is enough to show that $f$ is strictly monotonic.

Since $f'(x) = 1 + \dfrac{1}{x} > 0, \forall x \in D_f$, then $f$ is strictly increasing, so the point $c$ is unique.

**Theorem 134** ***(Generalized Intermediate Value Theorem)***
*Let $I$ be any interval of $\mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ be a continuous function. Let $x_1, x_2 \in I$ such that $x_1 < x_2$, then $\forall y_0 \in ]f(x_1), f(x_2)[ , \exists x_0 \in ]x_1, x_2[$ / $y_0 = f(x_0)$.*

**Theorem 135** *Let $I$ be any interval of $\mathbb{R}$ and let $f : I \longrightarrow \mathbb{R}$ be a continuous function, then $f(I)$ is an interval.*

**Remark 136** 1) *If $f$ is a continuous and increasing function on $[a, b]$, then $f([a, b]) = [f(a), f(b)]$.*

2) *If $f$ is a continuous and decreasing function on $[a, b]$, then $f([a, b]) = [f(b), f(a)]$.*

3) *In general $f([a, b]) \neq [f(a), f(b)]$.*

4) *If $f$ is a continuous function on $[a, b]$, then $f([a, b]) = [m, M]$, where $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$.*

### 3.4.5 Extension by continuity

The goal is to describe the extension of a function to a larger domain while maintaining its continuity.

**Definition 137** *Let $f : X \longrightarrow \mathbb{R}$ and $x_0 \notin X = D_f$ such that $f$ is defined in the neighborhood of $x_0$. We assume that $\lim\limits_{x\longrightarrow x_0} f(x) = \ell$ ($\ell$ finite).*

*Then, the function $\widetilde{f} : X \cup \{x_0\} \longrightarrow \mathbb{R}$ defined by*

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in X, \\ \ell, & x = x_0, \end{cases}$$

*is continuous at $x_0$.*
*$\widetilde{f}$ is said to be the extension of $f$ by continuity at point $x_0$.*

**Example 138** *Let $f(x) = \dfrac{\sin x}{x}$ be the function defined on $D_f = \mathbb{R}^*$.*

*Let $x_0 = 0 \notin D_f$.*

$\lim\limits_{x\longrightarrow 0} f(x) = \lim\limits_{x\longrightarrow x_0} \dfrac{\sin x}{x} = 1,$

*then $f$ admits an extension by continuity at the point $x_0 = 0$ and its extension is $\widetilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$\widetilde{f}(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

### 3.4.6 Uniform continuity of a function on an interval

**Definition 139** *Let $I$ be an interval and $f : I \longrightarrow \mathbb{R}$. We say that $f$ is uniformly continuous on $I$ if*

$$\forall \varepsilon > 0, \exists \alpha > 0, \forall x, x' \in I, (|x - x'| < \alpha \Longrightarrow |f(x) - f(x')| < \varepsilon).$$

**Remark 140** *Uniform continuity is a property of the function over the entire interval $I$, whereas continuity can be defined at a point $x_0$.*

**Example 141** 1) $f(x) = c, (c \in \mathbb{R})$ *is uniformly continuous on* $\mathbb{R}$.

2) $f(x) = x, (x \in \mathbb{R})$ *is uniformly continuous on* $\mathbb{R}$.

3) $f(x) = \sin x, (x \in \mathbb{R})$ *is uniformly continuous on* $\mathbb{R}$.

**Theorem 142** *If $f$ is uniformly continuous on $I$, then $f$ is continuous on $I$.*

**Remark 143** *The converse of this theorem is not true.*

**Example 144** *The function $f(x) = \dfrac{1}{x}$ is continuous on $]0,1]$, but it is not uniformly continuous on $]0,1]$.*

*Indeed, $f$ is not uniformly continuous on $]0,1] \Longleftrightarrow$*

$(\exists \varepsilon > 0, \forall \alpha > 0, \exists x, x' \in ]0,1], (|x - x'| < \alpha$ *and* $|f(x) - f(x')| \geq \varepsilon))$.

*Let $\alpha > 0$, we set $x = \dfrac{1}{n} \in ]0,1]$ and $x' = \dfrac{1}{2n} \in ]0,1]$,*

*we choose $n \in \mathbb{N}^*$ such that $|x - x'| < \alpha$,*

$$|x - x'| < \alpha \Longleftrightarrow \left|\frac{1}{n} - \frac{1}{2n}\right| < \alpha \Longleftrightarrow \frac{1}{2n} < \alpha \Longleftrightarrow n > \frac{1}{2\alpha},$$

*therefore, it is enough to take* $n = \left[\dfrac{1}{2\alpha}\right] + 1 \in \mathbb{N}^*$.

$|f(x) - f(x')| = |n - 2n| = n \geq 1 = \varepsilon$,

*so just take $\varepsilon = 1$.*

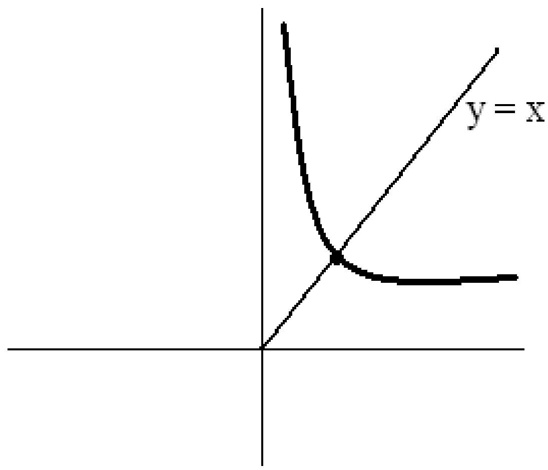
*Thus $f$ is not uniformly continuous on $]0,1]$.*

**Theorem 145** *If $f$ is continuous on a closed bounded interval $[a,b]$, then $f$ is uniformly continuous on $[a,b]$.*

**Theorem 146** ***(Fixed point Theorem)***

*Let $f : [a,b] \longrightarrow [a,b]$ be a continuous function, then there exists $x_0 \in [a,b]$ / $f(x_0) = x_0$.*

*$x_0$ is called a fixed point of $f$, and it is the abscissa of the point of intersection of the graph of $f$ with the first bisector $(y = x)$.*




### 3.4.7 Lipschitz function

**Definition 147** *Let $I$ be any interval and $f : I \longrightarrow \mathbb{R}$.*

*$f$ is called Lipschitzian if*

$$\exists k \geq 0, \forall x, x' \in I, |f(x) - f(x')| \leq k |x - x'| .$$

*If $0 \leq k < 1$ and $f : I \longrightarrow I$, we say that $f$ is contractive (or a contraction).*

**Remark 148** *Any Lipschitz function $f : I \longrightarrow I$ is uniformly continuous on $I$.*

*Indeed, let $\varepsilon > 0$,*

$\forall x, x' \in I, |f(x) - f(x')| \leq k |x - x'| \leq (k+1) |x - x'| < \varepsilon \Longrightarrow$

$|x - x'| < \dfrac{\varepsilon}{k+1} = \alpha,$

*therefore, it is enough to take $\alpha = \dfrac{\varepsilon}{k+1}$,*

*So $f$ is uniformly continuous on $I$.*

**Theorem 149** *Let $f : [a, b] \longrightarrow [a, b]$ be a contractive function, then $f$ admits a unique fixed point.*

### 3.4.8 Properties of monotonic functions on an interval

**Theorem 150** *Let $I$ be any interval and $f : I \longrightarrow \mathbb{R}$ a monotonic function on $I$, then*

$$f \text{ is continuous on } I \Longleftrightarrow f(I) \text{ is an interval.}$$

**Theorem 151** *Let $X \subset \mathbb{R}$ and $f$ a strictly monotonic fonction on $X$, then the function $f : X \longrightarrow f(X)$ is bijective and the inverse function $f^{-1} : f(X) \longrightarrow X$ is strictly monotonic on $f(X)$.*

**Theorem 152** ***(Inverse Function Theorem)***
*Let $I$ be any interval and $f : I \longrightarrow f(I)$ a continuous and strictly increasing function (or strictly decreasing, respectively). Then the inverse function $f^{-1} : f(I) \longrightarrow I$ is continuous and strictly increasing (or strictly decreasing, respectively).*

## 3.5 Exercises

**Exercise 153** *Let the function défined by :*

$$f(x) = \begin{cases} 0 & , \quad x \in \left]-\infty, 2\right], \\ a - \dfrac{b}{x} & , \quad x \in \left]2, 4\right], \\ 1 & , \quad x \in \left]4, +\infty\right[. \end{cases}$$

*Determine $a$ and $b$ so that $f$ is continuous on $\mathbb{R}$.*

**Solution** :
$f$ is continuous on $]-\infty, 2[$ because it is a constant function.

$f$ is continuous on $]2, 4[$ because it is the sum and the product of continuous functions on $]2, 4[$.

$f$ is continuous on $]4, +\infty[$ because it is a constant function.

The continuity at $x_0 = 2 :\ f(2) = 0$,

$$\lim_{x \overset{<}{\longrightarrow} 2} f(x) = \lim_{x \overset{>}{\longrightarrow} 2} f(x) = f(2) \Longleftrightarrow a - \frac{b}{2} = 0.$$

The continuity at $x_0 = 4 :\ f(4) = a - \dfrac{b}{4}$,

$$\lim_{x \overset{<}{\longrightarrow} 4} f(x) = \lim_{x \overset{>}{\longrightarrow} 4} f(x) = f(4) \Longleftrightarrow a - \frac{b}{4} = 1.$$

We solve the following system :

$$\begin{cases} a - \dfrac{b}{2} & = & 0, \\ a - \dfrac{b}{4} & = & 1. \end{cases} \Longrightarrow \begin{cases} a & = & 2, \\ b & = & 4. \end{cases}$$

**Exercise 154** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be a continuous function such that*

$f(a).f(b) < 0.$

*Prove that* $\forall x, y \in [a, b], \exists z \in [a, b] \ / \ f(z) = \dfrac{f(x) + f(y)}{3}$.

**Solution** :
$f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function such that $f(a).f(b) < 0$,

therefore, by the intermediate value theorem, $\exists c \in \, ]a, b[ \ / \ f(c) = 0$.

Furthermore, $f$ is continuous on a closed and bounded interval, so it is bounded and,

moreover, it attains its bounds,

i.e. $\exists x_1, x_2 \in [a, b] \ / \ f(x_1) = \inf\limits_{x\in[a,b]} f(x) = m$ and $f(x_2) = \sup\limits_{x\in[a,b]} f(x) = M$.

$x \in [a, b] \Longrightarrow m \leq f(x) \leq M$,

$y \in [a, b] \Longrightarrow m \leq f(y) \leq M$,

$c \in \, ]a, b[ \Longrightarrow m \leq f(c) = 0 \leq M$.

By adding the three inequalities, we obtain

$3m \leq f(x) + f(y) \leq 3M \Longrightarrow m \leq \dfrac{f(x) + f(y)}{3} \leq M$,

therefore, by the generalized intermediate value theorem, $\exists \, z \in [a, b] \ / \ f(z) = \dfrac{f(x) + f(y)}{3}$.

**Exercise 155** 1) *Using the intermediate value theorem, show that the equation* : $xe^{\sin x} = \cos x$ *admit a solution in* $]0, \dfrac{\pi}{2}[$.

2) *Prove that this solution is unique.*

**Solution** :
1) $xe^{\sin x} = \cos x \Longleftrightarrow xe^{\sin x} - \cos x = 0.$

We set $f(x) = xe^{\sin x} - \cos x$,

$f$ is continuous on $\left[0, \frac{\pi}{2}\right]$.
$f(0) = -1 < 0$ and $f(\frac{\pi}{2}) = \frac{\pi}{2}e > 0$,

therefore, by the intermediate value theorem, $\exists c \in ]0, \frac{\pi}{2}[\ /\ f(c) = 0.$

2) Let's show that this solution is unique :

It is enough to show that the function is strictly monotone.

$f'(x) = e^{\sin x} + x\cos x.e^{\sin x} + \sin x > 0 \quad \text{on } \left[0, \frac{\pi}{2}\right]$,

then $f$ is strictly increasing, hence the point $c$ is unique.

**Exercise 156** *Let the function be defined by* $f(x) = \dfrac{1-\cos x}{x^2}$.

1) *Determine the domain of definition* $D_f$ *of the function* $f$.

2) *Study the continuity of* $f$ *on* $D_f$.

3) *Study the extension by continuity of* $f$ *on* $\mathbb{R}$.

**Solution** :
Let the function definie by $f(x) = \dfrac{1-\cos x}{x^2}$.

1) $D_f = \mathbb{R}^*$.

2) $f$ is continuous on $\mathbb{R}^*$ because it is the sum and the product of continuous functions on $\mathbb{R}^*$.

3) The extension by continuity of $f$ on $\mathbb{R}$ : $0 \notin D_f$,

$$\lim_{x\longrightarrow 0} f(x) = \lim_{x\longrightarrow 0} \frac{1-\cos x}{x^2} = \lim_{x\longrightarrow 0} \frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} = \lim_{x\longrightarrow 0} \frac{1-\cos^2 x}{x^2(1+\cos x)},$$

$$\lim_{x\longrightarrow 0} f(x) = \lim_{x\longrightarrow 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim_{x\longrightarrow 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{(1+\cos x)} = \frac{1}{2},$$

since $\lim\limits_{x\longrightarrow 0} \dfrac{\sin x}{x} = 1.$

Therefore $f$ admits an extension by continuity at 0, and its extension is

$$\widetilde{f}(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0, \\ \dfrac{1}{2}, & x = 0. \end{cases}$$

# Chapter 4

# Derivability of functions of one real variable

## 4.1 Generalities

### 4.1.1 Definition of the derivability

**Definition 157** *Let $I$ be an interval, $x_0 \in I$ and $f : I \longrightarrow \mathbb{R}$ a real function.*

*We say that $f$ is derivable at the point $x_0$ if $\lim\limits_{x \longrightarrow x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists (finite).*

*This limit is called the derivative of $f$ at the point $x_0$, it is unique and is denoted by $f'(x_0)$ or $\dfrac{df}{dx}(x_0)$.*

**Remark 158** *- The function $f$ is derivable on $I$ if it is derivable at every point $x_0 \in I$.*

*- The function $f' : I \longrightarrow \mathbb{R}$ is called the derivative of the function $f$.*

*- We can also write:*

$\dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + \varepsilon(x)$, *with* $\lim\limits_{x \longrightarrow x_0} \varepsilon(x) = 0$,

*then,* $f(x) - f(x_0) = (f'(x_0) + \varepsilon(x))(x - x_0)$,

*so we deduce that* $f(x) = f(x_0) + f'(x_0)(x - x_0) + (x - x_0)\varepsilon(x)$.

*- If we set* $h = x - x_0$,

*when $x$ tends to $x_0$, then $h$ tends to $0$,*

*we get,* $f'(x_0) = \lim\limits_{x \longrightarrow x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim\limits_{h \longrightarrow 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$,

*and we also obtain*

$f(x_0+h)-f(x_0)=hf'(x_0)+h\varepsilon(h),$ *with* $\lim_{h\longrightarrow 0}\varepsilon(h)=0.$

*- then $f$ is derivable at the point $x_0$ if and only if there exists $\ell\in\mathbb{R}$ and a function $\varepsilon$ such that*

$\forall x_0+h\in I,\ f(x_0+h)=f(x_0)+h\ell+h\varepsilon(h),$ *with* $\lim_{h\longrightarrow 0}\varepsilon(h)=0,$

*i.e.* $f(x)=f(x_0)+(x-x_0)\ell+(x-x_0)\varepsilon(x),$ *with* $\lim_{x\longrightarrow x_0}\varepsilon(x)=0.$

**Example 159** 1) $f(x)=c,\forall x\in\mathbb{R},$

$$\forall x_0\in\mathbb{R}, f'(x_0)=\lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{x-x_0}=0,$$

*then* $\forall x\in\mathbb{R},\ f'(x)=0.$

2) $f(x)=x,\forall x\in\mathbb{R},$

$$\forall x_0\in\mathbb{R}, f'(x_0)=\lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\longrightarrow x_0}\frac{x-x_0}{x-x_0}=1,$$

*so* $\forall x\in\mathbb{R},\ f'(x)=1.$

3) $f(x)=x^2,\forall x\in\mathbb{R},$

$$f'(x_0)=\lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\longrightarrow x_0}\frac{x^2-x_0^2}{x-x_0}=\lim_{x\longrightarrow x_0}\frac{(x-x_0)(x+x_0)}{x-x_0}=2x_0,$$

*hence* $\forall x\in\mathbb{R},\ f'(x)=2x.$

4) $f(x)=\sin x,\forall x\in\mathbb{R},$

$$f'(x_0)=\lim_{x\longrightarrow x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\longrightarrow x_0}\frac{\sin x-\sin x_0}{x-x_0}=\lim_{x\longrightarrow x_0}\frac{2\sin(\frac{x-x_0}{2})\cos(\frac{x+x_0}{2})}{x-x_0},$$

$$=\lim_{x\longrightarrow x_0}\frac{\sin(\frac{x-x_0}{2})}{\frac{x-x_0}{2}}\cos(\tfrac{x+x_0}{2})=\cos x_0,$$

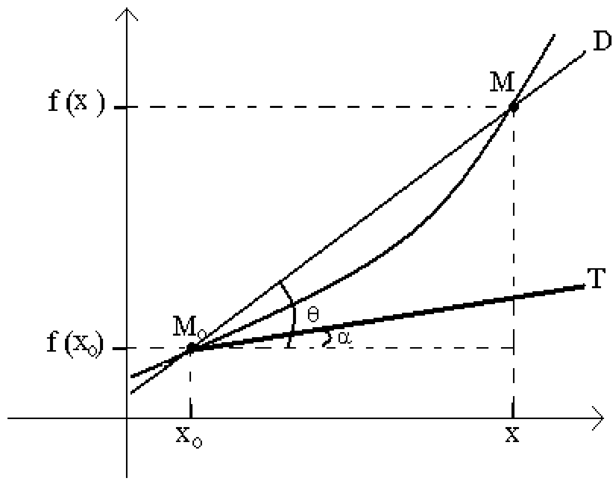
*therefore* $\forall x\in\mathbb{R},\ f'(x)=\cos x.$

### 4.1.2 Geometric interpretation

Let $f$ be a derivable function at the point $x_0$.

The line with equation : $(T): y=f(x_0)+f'(x_0)(x-x_0)$, is called the tangent to the graph of $f$ at the point $M_0(x_0,f(x_0))$.

This line intersects the graph at a single point $M_0(x_0,f(x_0))$.
$\tan\theta=\dfrac{f(x)-f(x_0)}{x-x_0}$.




$D$ is the line that passes through the points $M_0(x_0, f(x_0))$ and $M(x, f(x))$.

Then $\tan\theta$ is the slope of the line $D$ (or the slope coefficient of $D$).

When $x$ tends to $x_0$, we notice that the line $D$ tends to the line $(T)$ and therefore, $\tan\theta$ tends to the slope of $(T)$ which is the derivative of $f$ at the point $x_0$.

Then we have $f'(x_0) = \tan\alpha$,

where $\alpha$ denotes the angle formed by the axis $(OX)$ and the tangent to the curve of $f$ at the point $M_0$.

**Conclusion**
The derivative of $f$ at $x_0$is the slope of the tangent to the curve of $f$ at the point $M_0(x_0, f(x_0))$.

**Remark 160** *If* $f'(x_0) = 0$,*then* $\tan\alpha = 0 \Longrightarrow \alpha = 0$,

*so* $(T)$ *the tangent to the curve of* $f$ *is horizontal at the point* $M_0(x_0, f(x_0))$.

### 4.1.3 Right and left derivability of the function at $x_0$

Let $f : I \longrightarrow \mathbb{R}$ be a function and $x_0 \in I$.

$f$ is derivable from the right at $x_0$ if $f'_d(x_0) = \lim\limits_{x \overset{>}{\longrightarrow} x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists (finite).

$f$ is derivable from the left at $x_0$ if $f'_g(x_0) = \lim\limits_{x \overset{<}{\longrightarrow} x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists (finite).

**Conclusion**

$f$ is derivable at $x_0 \Longleftrightarrow f$ is derivable from the right and from the left at $x_0$ and $f'_d(x_0) = f'_g(x_0)$

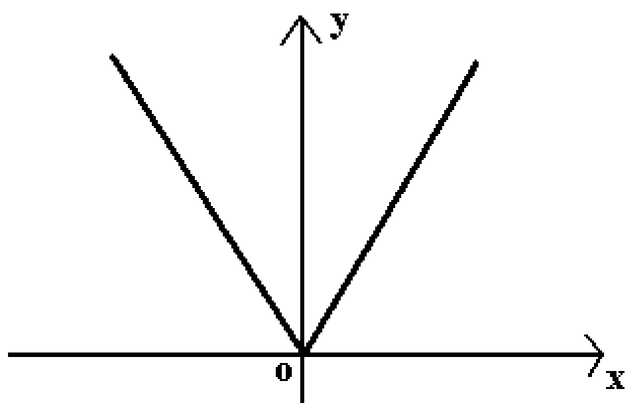
**Example 161** *Let the function* $f(x) = |x|, \forall x \in \mathbb{R}$.

*We study the derivability of* $f$ *at the point* $x_0 = 0$ :

$f'_d(0) = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{|x|}{x} = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{x}{x} = 1,$

$f'_g(0) = \lim\limits_{x \overset{<}{\longrightarrow} 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \overset{<}{\longrightarrow} 0} \dfrac{|x|}{x} = \lim\limits_{x \overset{<}{\longrightarrow} 0} \dfrac{-x}{x} = -1,$

*since* $f'_d(0) \neq f'_g(0)$, *then* $f$ *is not derivable at* $x_0 = 0$.




**Example 162** *Let the function* $f(x) = \sqrt{x}, \forall x \geq 0$.

*We study the derivability of* $f$ *at the point* $x_0 = 0$ :

$f'_d(0) = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{\sqrt{x}}{x} = \lim\limits_{x \overset{>}{\longrightarrow} 0} \dfrac{1}{\sqrt{x}} = +\infty,$

*then* $f$ *is not derivable at the point* $x_0 = 0$.

## 4.2 Properties of derivable functions

### 4.2.1 Derivability and continuity

**Theorem 163** *Let* $f : I \longrightarrow \mathbb{R}$ *be a function and* $x_0 \in I$.

*If* $f$ *is derivable at* $x_0$, *then* $f$ *is continuous at* $x_0$.

**Remark 164** 1) *The contrapositive of this implication :*

*If $f$ is not continuous at $x_0 \Longrightarrow f$ is not derivable at $x_0$.*

2) *The converse of this theorem is not true.*

**Example 165** *The function $f(x) = |x|$ is continuous at $0$, but it is not derivable at $0$.*

### 4.2.2 Operations on derivable fonctions

**Theorem 166** *Let $f, g : I \longrightarrow \mathbb{R}$ be two derivable functions at $x_0 \in I$, then*

1) *$(f + g)$ is derivable at $x_0$ and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.*

2) *$\forall \lambda \in \mathbb{R}, \lambda f$ is derivable at $x_0$ and $(\lambda f)'(x_0) = \lambda f'(x_0)$.*

3) *$(f.g)$ is derivable at $x_0$ and $(f.g)'(x_0) = (f'.g)(x_0) + (f.g')(x_0)$.*

4) *If $g(x_0) \neq 0, \dfrac{f}{g}$ is derivable at $x_0$ and $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$.*

### 4.2.3 Derivative of a composed function

**Theorem 167** *Let $f$ a function defined on the interval $I$, $g$ a function defined on the interval $J$ such that $f(I) \subset J$ and $x_0 \in I$. If $f$ is derivable at $x_0$ and $g$ derivable at $f(x_0)$, then $(g \circ f)$ is derivable at $x_0$ and we have*

$$(g \circ f)'(x_0) = f'(x_0).g'(f(x_0)).$$

**Example 168** 1) $f(x) = \sin(x^2) \Longrightarrow f'(x) = 2x\cos(x^2),\ \forall x \in \mathbb{R}$.

2) $f(x) = \ln \sqrt{x} \Longrightarrow f'(x) = \dfrac{1}{2\sqrt{x}} \dfrac{1}{\sqrt{x}} = \dfrac{1}{2x},\ \forall x > 0$.

**Remark 169** - *The derivative of an even function is an odd function.*

*Indeed, let $f$ be an even function $\Longrightarrow f(-x) = f(x) \Longrightarrow -f'(-x) = f'(x) \Longrightarrow f'$ is odd.*

- *The derivative of an odd function is an even function.*
*Indeed, let $f$ be an odd function $\Longrightarrow f(-x) = -f(x) \Longrightarrow -f'(-x) = -f'(x) \Longrightarrow f'$ is even.*

**Example 170** *The function $f(x) = \sin x$ is odd $\Longrightarrow f'(x) = \cos x$ is even.*

### 4.2.4 Derivative of the reciprocal function

**Theorem 171** *Let $f$ be a bijective and continuous function from an interval $I$ to an interval $J$ and derivable at $x_0 \in I$ such that $f'(x_0) \neq 0$.*

*Then the reciprocal function $f^{-1} : J \longrightarrow I$ is derivable at $y_0 = f(x_0)$ and we have*

$$\left(f^{-1}\right)'(f(x_0)) = \frac{1}{f'(x_0)}$$

**Example 172** *The function* $\tan x : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[ \longrightarrow \mathbb{R}$ *is a bijective function,*

*Thus, it has a reciprocal function :* $\arctan x : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

$\forall x \in \mathbb{R}, (\arctan x)' = \dfrac{1}{(\tan y)'} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$, *with* $x = \tan y$

### 4.2.5 Derivatives of order higher than 1

- If $f$ is derivable, its derivative $f'$ is called the first derivative of $f$.

- If $f'$ is derivable, its derivative $f''$ is called the second derivative (or of order 2) of $f$.

- We define by recurrence the successive derivatives of $f$. Thus $f^{(n)}$ is the $nth$ derivative or the $nth$-order derivative of $f$, it is the derivative of the function $f^{(n-1)}$, i.e. $f^{(n)}(x) = (f^{(n-1)})'(x)$.

**Convention** : $f^{(0)} = f$.

**Example 173** *We show by recurrence that :*

$$\forall n \in \mathbb{N}, (\sin x)^{(n)} = \sin(x + n\frac{\pi}{2})$$

*Indeed,*
*for* $n = 0 : (\sin x)^{(0)} = \sin x = \sin(x + 0\frac{\pi}{2})$,

*for* $n = 1 : (\sin x)^{(1)} = \cos x = \sin(x + \frac{\pi}{2})$.

*We assume that the property is true up to order $n$ and we show that it is true for order $(n+1)$.*

*We assume that* $(\sin x)^{(n)} = \sin(x + n\frac{\pi}{2})$, *and we show that* $(\sin x)^{(n+1)} = \sin(x + (n+1)\frac{\pi}{2})$?

$$(\sin x)^{(n+1)} = \left((\sin x)^{(n)}\right)' = (\sin(x + n\tfrac{\pi}{2}))' = \cos(x + n\tfrac{\pi}{2})$$
$$= \sin(x + n\tfrac{\pi}{2} + \tfrac{\pi}{2}) = \sin(x + (n+1)\tfrac{\pi}{2}).$$

*Then,* $\forall n \in \mathbb{N}, (\sin x)^{(n)} = \sin(x + n\frac{\pi}{2})$.

*We show in the same way that*

$$\forall n \in \mathbb{N}, (\cos x)^{(n)} = \cos(x + n\frac{\pi}{2})$$

### 4.2.6 Functions of class $C^n$

**Definition 174** *Let $f : I \longrightarrow \mathbb{R}$ be a function and $n \in \mathbb{N}^*$. We say that $f$ is of class $C^n$ (or $n$ times continuously derivable) if it is $n$ times derivable and if $f^{(n)}$ is continuous on $I$.*

**Remark 175** $C^n$ $(I)$ : *it is the set of functions of class $C^n$ on $I$.*

$C^\infty$ $(I)$ : *It is the set of functions that are infinitely derivable on $I$.*

$C^0$ $(I) = C(I)$ : *it is the set of continuous functions on $I$.*

**Example 176** 1) $f(x) = x^2$,

$f'(x) = 2x$, $f''(x) = 2$, $f^{(3)}(x) = 0, ......,$ *then* $f \in C^\infty$ $(\mathbb{R})$.

2) $f(x) = \sin x$,

$f^{(n)}(x) = \sin(x + n\frac{\pi}{2})$ *is continuous, then* $f \in C^\infty$ $(\mathbb{R})$.

### 4.2.7 Leibniz Formula

**Theorem 177** *Let $f, g : I \longrightarrow \mathbb{R}$ and $x_0 \in I$ such that $f^{(n)}(x_0)$ and $g^{(n)}(x_0)$ exist $(n \in \mathbb{N}^*)$, then the function $f.g$ admits a nth derivative at point $x_0$, and we have*

$$(f.g)^{(n)}(x_0) = \sum_{k=0}^{n} C_n^k f^{(k)}(x_0).g^{(n-k)}(x_0), \quad \text{where } C_n^k = \frac{n!}{k!(n-k)!}.$$

**Example 178** *Calculate the nth derivative of the function $f(x) = x^2 \sin x$.*

$f^{(n)}(x) = (x^2 \sin x)^{(n)} = \sum\limits_{k=0}^{n} C_n^k (x^2)^{(k)}.(\sin x)^{(n-k)} = C_n^0(x^2).(\sin x)^{(n)}$

$+C_n^1(x^2)'.(\sin x)^{(n-1)} + C_n^2(x^2)''.(\sin x)^{(n-2)} + 0 + 0...$

$= x^2 \sin(x + n\frac{\pi}{2}) + n2x \sin(x + (n-1)\frac{\pi}{2}) + n(n-1)\sin(x + (n-2)\frac{\pi}{2}).$

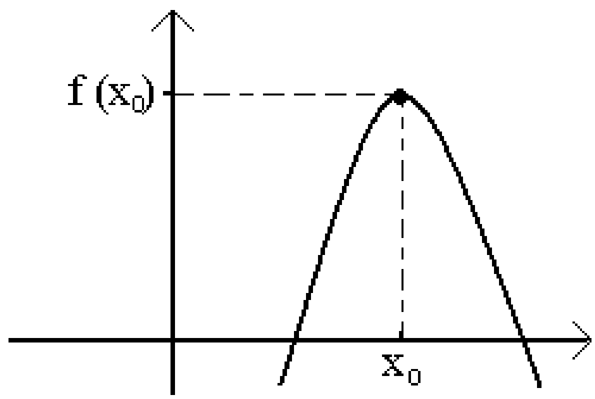
## 4.3 Main theorems

### 4.3.1 Maximum and minimum

**Definition 179** ***(Local maximum)***
*Let $f : I \longrightarrow \mathbb{R}$ and $x_0 \in I$.*

*We say that $f$ admits a local maximum at $x_0$ if*

$$\exists \alpha > 0, \forall x \in I, |x - x_0| < \alpha \Longrightarrow f(x) \leq f(x_0).$$




**Definition 180** ***(Local Minimum)***
*Let $f : I \longrightarrow \mathbb{R}$ and $x_0 \in I$.*

*We say that $f$ admits a local minimum at $x_0$ if*

$$\exists \alpha > 0, \forall x \in I, |x - x_0| < \alpha \Longrightarrow f(x) \geq f(x_0).$$

In both cases, we say that $f$ admits a local extremum at the point $x_0$.

**Theorem 181** *Let $f : I \longrightarrow \mathbb{R}$ and $x_0 \in I$.*

*If $f$ admits an extremum (maximum or minimum) at the point $x_0$ and if $f'(x_0)$ exists, then $f'(x_0) = 0$.*

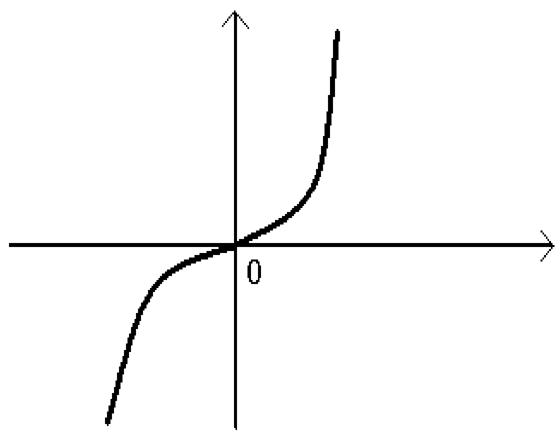
**Remark 182** *A function can admit an extremum at $x_0$, without being derivable at $x_0$.*

**Example 183** *$f(x) = |x|$ admits a minimum at $x_0 = 0$, whereas it is not derivable at $x_0 = 0$.*

**Remark 184** *The converse of the theorem is false.*

**Example 185** *Let the function $f(x) = x^3$, $\forall x \in \mathbb{R}$.*

*We have $f'(0) = 0$, but $f$ doesn't admit an extremum at $0$.*




### 4.3.2 Rolle's Theorem

**Theorem 186** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be the function such that*

1) *$f$ is continuous on $[a, b]$,*

2) *$f$ is derivable on $]a, b[$,*

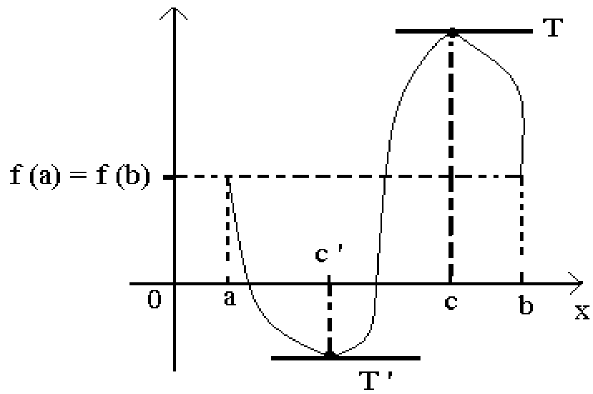
3) *$f(a) = f(b)$,*

*then, there exists $c \in\, ]a, b[\ /\ f'(c) = 0$.*

**Geometric interpretation of Rolle's Theorem :**

$\exists c \in\, ]a, b[\ /\ f'(c) = 0$, so there exists at least one tangent to the horizontal curve of f in the interval $]a, b[$.

**Example 187** *The function $f(x) = x^2$, $x \in [-1, 1]$, verifies all the conditions of Rolle's Theorem, so there exists $c \in\, ]-1, 1[\ /\ f'(c) = 0$.*




**Remark 188** *All the assumptions of Rolle's Theorem are essential for the application of Rolle's Theorem.*

**Example 189** *The function* $f(x) = |x|$ *verifies all the conditions of Rolle's theorem except the derivability in* $0$. *We notice that there is no point* $c \in \left]-1, 1\right[$ *such that* $f'(c) = 0$.

**Remark 190** *The converse of Rolle's Theorem is not true.*

**Example 191** *The function* $f(x) = x^3$ *verifies* $f'(0) = 0$, *while it does not satisfy all the hypotheses of Rolle's Theorem on* $[-1, 1]$ $(f(1) \neq f(-1))$.

### 4.3.3 The Mean Value Theorem (M.V.T)

**Theorem 192** *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be the function such that*

1) $f$ *is continuous on* $[a, b]$,

2) $f$ *is derivable on* $\left]a, b\right[$,

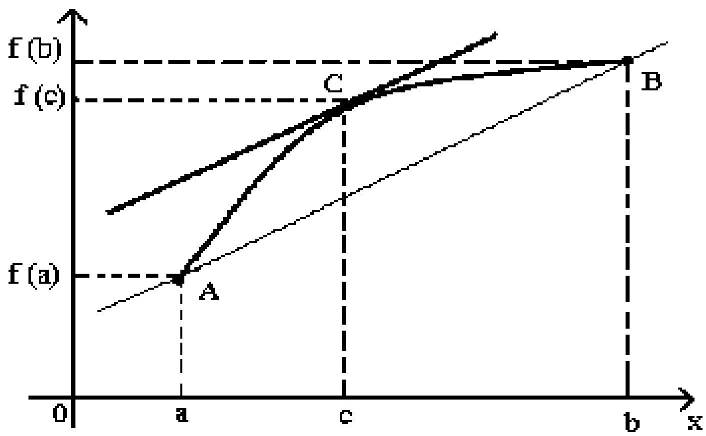
*then there exists* $c \in \left]a, b\right[\ /\ f(b) - f(a) = (b - a) f'(c)$.

**Geometric interpretation of the Mean Value Theorem (M.V.T) :**

We have $\dfrac{f(b) - f(a)}{b - a} = f'(c)$,

then $f'(c)$ is the slope of the line $(AB)$ passing through the points $A(a, f(a))$ and $B(b, f(b))$,

from where there exists a point $C(c, f(c))$ such that $(T)$ the tangent to the curve of $f$ at this point is parallel to the line $(AB)$, because $(T)$ and $(AB)$ have the same slope.




**Corollary 193** *Let $I$ be any interval and $f : I \longrightarrow \mathbb{R}$ be a derivable function on $I$ and let $x_1, x_2 \in I$ with $x_1 < x_2$. Then, there exists $c \in ]x_1, x_2[$ such that*

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c).$$

**Another formulation of the Mean Value Theorem (M.V.T).**
We set $h = b - a > 0$,

$$c \in ]a, b[ \Longrightarrow c = a + (c - a) = a + \frac{(c-a)}{h}h = a + \theta h, \text{ with } \theta \in ]0, 1[\,.$$

With this notation, the Mean Value Theorem formula can be written as :

$$\exists \theta \in ]0, 1[ \ / \ f(a + h) - f(a) = hf'(a + \theta h).$$

**Example 194** *Prove that* $\forall x \in \left[0, \frac{\pi}{2}\right[, \tan x \geq x.$

- *If* $x = 0$, $\tan 0 = 0 \geq 0$,

- *If* $x \in \left]0, \frac{\pi}{2}\right[$, *we set* $f(t) = \tan t$, *and* $[a, b] = [0, x]$, *we have*

*$f$ is continuous on $[0, x]$ and derivable on $]0, x[$, then according to the Mean Value Theorem (M.V.T), there exists $c \in ]0, x[ \ / \ f(x) - f(0) = (x - 0)f'(c)$,*

*then,* $\tan x - \tan 0 = x(1 + \tan^2 c) \geq x \Longrightarrow \tan x \geq x.$

*Hence* $\forall x \in \left[0, \frac{\pi}{2}\right[, \tan x \geq x.$

### 4.3.4 Application of the Mean Value Theorem (M.V.T) to the variations of functions

**Proposition 195** *Let $f : I \longrightarrow \mathbb{R}$ be a continuous and derivable function on $I$ (any interval). Then we have*

1) $\forall x \in I,\ f'(x) = 0 \iff f$ *is a constant function.*

2) $\forall x \in I,\ f'(x) \geq 0 \iff f$ *is an increasing fonction.*

3) $\forall x \in I,\ f'(x) \leq 0 \iff f$ *is a decreasing function.*

### 4.3.5 Generalized Mean Value Theorem (G.M.V.T)

**Theorem 196** *Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two functions continuous on $[a, b]$ and derivable on $]a, b[$ . If $\forall x \in\ ]a, b[\ , g'(x) \neq 0$, then we have*

$$\exists c \in\ ]a, b[\ /\ \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'c)}{g'(c)}.$$

**Remark 197** *From this theorem, we deduce L'Hôpital's rule, which allows us to compute limits of the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.*

### 4.3.6 L'Hôpital's rule

**Theorem 198** *Let $I$ be any interval of $\mathbb{R}$, $x_0 \in I$ and two functions $f, g : I \longrightarrow \mathbb{R}$ continuous on $I$ and derivable on $I \smallsetminus \{x_0\}$.*

*If $f(x_0) = g(x_0) = 0$ and $\forall x \in I \smallsetminus \{x_0\}, g'(x) \neq 0$, then*

*if $\displaystyle\lim_{x \longrightarrow x_0} \frac{f'(x)}{g'(x)} = \ell$ (finite or infinite), then $\displaystyle\lim_{x \longrightarrow x_0} \frac{f(x)}{g(x)} = \ell$.*

**Remark 199** *L'Hôpital's rule can be applied in the following cases:*

1) *$x_0$ is an adherent point of $I$.*

2) *The hypothesis $f(x_0) = g(x_0) = 0$ is replaced by $\displaystyle\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) =$* 0.

3) *In the case $x_0 = \infty$,* $\displaystyle\lim_{x \longrightarrow x_0} f(x) = \lim_{x \longrightarrow x_0} g(x) = \infty$.

**Example 200** *Calculate $\displaystyle\lim_{x \longrightarrow 0} \frac{1 - \cos x}{x^2}$.*

*Since the assumptions are satisfied and $\displaystyle\lim_{x \longrightarrow 0} \frac{f'(x)}{g'(x)} = \lim_{x \longrightarrow 0} \frac{(1 - \cos x)'}{(x^2)'} = \lim_{x \longrightarrow 0} \frac{\sin x}{2x} = \frac{1}{2}$, then $\displaystyle\lim_{x \longrightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.*

**Remark 201** *The converse of L'Hôpital's rule is not true, i.e. if* $\lim\limits_{x\longrightarrow x_0}\dfrac{f(x)}{g(x)}$ *exist, this does not imply that* $\lim\limits_{x\longrightarrow x_0}\dfrac{f'(x)}{g'(x)}$ *exists.*

**Example 202** *Let the functions defined by*

$$f(x)=\begin{cases} x^2\sin\dfrac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases} \quad et\ g(x)=\sin x$$

**Example 203** $\lim\limits_{x\longrightarrow 0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\longrightarrow 0}\dfrac{x^2\sin\frac{1}{x}}{\sin x}=\lim\limits_{x\longrightarrow 0}\left(\dfrac{x}{\sin x}\right)\left(x\sin\dfrac{1}{x}\right)=1.0=0$,

*while* $\lim\limits_{x\longrightarrow 0}\dfrac{f'(x)}{g'(x)}=\lim\limits_{x\longrightarrow 0}\dfrac{1}{\cos x}(2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x})$ : *doesn"t exist,*

*because* $\lim\limits_{x\longrightarrow 0}\cos\dfrac{1}{x}$ *doesn't exist.*

## 4.4 Taylor's formula

A function $f$ continuous son$[a,b]$ and derivable at $x_0\in\,]a,b[$ can be written in the vicinity of $x_0$ in the following form :

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+(x-x_0)\varepsilon(x)\ /\ \lim_{x\longrightarrow x_0}\varepsilon(x)=0.$$

This means that $f$ can be approximated by the polynomial of degree 1 :

$P(x)=f(x_0)+f'(x_0)(x-x_0)$,

$R(x)=(x-x_0)\varepsilon(x)$ : it is the error made by this approximation.

Taylor's formula generalizes this result by showing that functions that are $n$-times deriviable can be approximated in the vicinity of $x_0$ by polynomials of degree $n$, i.e.

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+.......+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$$

where $P_n(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+.......+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ is the polynomial of degree that approximates $f$ with an accuracy equal to $R_n(x)$.

The error $R_n(x)$ is called the remainder of order $n$ and has several forms depending on the derivability conditions imposed on $f$, which gives us several forms of Taylor's formula.

### 4.4.1 Taylor's formula with Lagrange remainder

**Theorem 204** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a function such that $f \in C^n([a,b])$ and $f^{(n)}$ derivable on $]a,b[$;. Then there exists $c \in ]a,b[$ such that*

$$\begin{aligned} f(b) &= f(a) + \frac{(b-a)}{1!}f'(a) + \frac{(b-a)^2}{2!}f''(a) + ....... + \frac{(b-a)^n}{n!}f^{(n)}(a) \\ &+ \frac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(c). \end{aligned}$$

*It is the Taylor formula with Lagrange's remainder :* $R_n = \dfrac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(c).$

### 4.4.2 Taylor-Maclaurin formula

If $h = b - a$, then $c = a + \theta h$ with $\theta \in ]0,1[$ and if we substitute into the Taylor-Lagrange formula, we obtain the Maclaurin Taylor formula.

**Theorem 205** *Let $I$ an interval of $\mathbb{R}$, $a \in I$, $f \in C^n(I)$ and $f$ admits a derivative of order $(n+1)$ on $I$. Then for any $a + h \in I$, there exists $\theta \in ]0,1[$ such that*

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + ....... + \frac{h^n}{n!}f^{(n)}(a) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta h).$$

**Remark 206** *If we set $a = 0$ and $x = a + h$ By substituting into the Taylor-Maclaurin formula, we obtain the Maclaurin formula of order n with Lagrange's remainder.: $\forall x \in I, \exists \theta \in ]0,1[$ such that*

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + ..... + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\theta x).$$

### 4.4.3 Taylor's formula with Young's remainder

**Theorem 207** *Let $f : [a,b] \longrightarrow \mathbb{R}, x_0 \in [a,b]$ and assume that $f^{(n)}(x_0)$ exists (finite). Then for any $x \in [a,b]$, we have*

$$\begin{aligned} f(x) &= f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + ..... + \frac{(x-x_0)^n}{n!}f^{(n)}(x_0) \\ &+ o((x-x_0)^n), \end{aligned}$$

*such that* $\lim\limits_{x\longrightarrow x_0} \dfrac{o((x-x_0)^n)}{(x-x_0)^n} = 0.$

*The remainder* $o((x-x_0)^n) = (x-x_0)^n \varepsilon(x)\ /\ \lim\limits_{x\longrightarrow x_0} \varepsilon(x) = 0.$

### 4.4.4 Maclaurin-Young formula

By taking $x_0 = 0$ and with the same assumptions as those of the Taylor-Young formula, we obtain the Maclaurin formula of order $n$ with the following Young's remainder term :

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \ldots.. + \frac{x^n}{n!}f^{(n)}(0) + x^n\varepsilon(x)\ /\ \lim_{x\longrightarrow 0}\varepsilon(x) = 0.$$

**Example 208** 1) $\forall x \in \mathbb{R}, e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \ldots.. + \dfrac{x^n}{n!} + o(x^n)\ /\ \lim\limits_{x\longrightarrow 0}\dfrac{o(x^n)}{x^n} = 0.$

2) $\forall x \in \mathbb{R}, \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots.. + (-1)^n\dfrac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})\ /$ $\lim\limits_{x\longrightarrow 0}\dfrac{o(x^{2n+1})}{x^{2n+1}} = 0.$

3) $\forall x \in \mathbb{R}, \cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots.. + (-1)^n\dfrac{x^{2n}}{(2n)!} + o(x^{2n})\ /\ \lim\limits_{x\longrightarrow 0}\dfrac{o(x^{2n})}{x^{2n}} = 0.$

4) $f(x) = (1+x)^\alpha,\ \ x \in ]-1, +\infty[,\ \ \forall \alpha \in \mathbb{R},$

*$f$ is indefinitely derivable :*

$\forall k \in \mathbb{N}^*,\ \ f^{(k)}(x) = \alpha(\alpha-1)\ldots..(\alpha-k+1)(1+x)^{\alpha-k},$

*then* $f^{(k)}(0) = \alpha(\alpha-1)\ldots..(\alpha-k+1)$

$(1+x)^\alpha = 1 + \dfrac{\alpha x}{1!} + \dfrac{\alpha(\alpha-1)}{2!}x^2 + \ldots.. + \dfrac{\alpha(\alpha-1)\ldots.(\alpha-n+1)}{n!}x^n + o(x^n)$ $/\ \lim\limits_{x\longrightarrow 0}\dfrac{o(x^{2n+1})}{x^{2n+1}} = 0.$

*If* $\alpha = -1$ :

$\dfrac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 + \ldots. + (-1)^n x^n + o(x^n),\ /\ \lim\limits_{x\longrightarrow 0}\dfrac{o(x^n)}{x^n} = 0,$

*we deduce*

$\dfrac{1}{1-x} = \dfrac{1}{1+(-x)} = 1 + x + x^2 + \ldots. + x^n + o(x^n),\ /\ \lim\limits_{x\longrightarrow 0}\dfrac{o(x^n)}{x^n} = 0.$

## 4.5 Exercises

**Exercise 209** *We define the function* $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , \quad x \neq 0, \\ 0 & , \quad x = 0. \end{cases}$

1) *Give the domain of definition* $D_f$ *of the function* $f$.

2) *Study the continuity and derivability of* $f$ *on* $D_f$ *and calculate* $f'$.

3) $f$ *is it of class* $C^1(\mathbb{R})$?

4) *Prove that :* $\forall x \in \mathbb{R}, \ \ |\sin x| \leq |x|$.

**Solution :**

We define the function $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , \quad x \neq 0, \\ 0 & , \quad x = 0. \end{cases}$

1) The domain of definition $Df = \mathbb{R}$.

2) Study the continuity and derivability of $f$ on $D_f$ :

$f$ is continuous and derivable on $\mathbb{R}^*$ because it is the product and the composition of continuous and deriviable functions on $\mathbb{R}^*$.

Continuity at 0 : $f(0) = 0$,

$\lim\limits_{x \longrightarrow 0} f(x) = \lim\limits_{x \longrightarrow 0} x^2 \sin \dfrac{1}{x} = 0 = f(0)$,

(since $\lim\limits_{x \longrightarrow 0} x^2 = 0$ and $\sin \dfrac{1}{x}$ is bounded).

Then $f$ is continuous at 0, hence $f$ is continuous on $\mathbb{R}$.

Derivability at 0 :

$$\lim_{x \longrightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \longrightarrow 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{n \longrightarrow 0} x \sin \frac{1}{x} = 0,$$

(since $\lim\limits_{x \longrightarrow 0} x = 0$ and $\sin \dfrac{1}{x}$ is bounded).

Then $f$ is derivable at 0 and $f'(0) = 0$.

Thus $f$ is continuous and derivable on $\mathbb{R}$.

$$f'(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

3) $f$ is of class $C^1(\mathbb{R})$ if $f'$ is continuous on $\mathbb{R}$.

$f'$ is continuous on $\mathbb{R}^*$ because it is the sum, the product and the composition of continuous functions on $\mathbb{R}^*$.

Continuity of $f'$ at $0:$ $f'(0)=0$,

$\lim\limits_{x\longrightarrow 0} f'(x) = \lim\limits_{x\longrightarrow 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right):$ does not exist (since $\lim\limits_{x\longrightarrow 0}\cos\frac{1}{x}$ does not exist).

Then $f'$ is not continuous at 0, therefore $f \notin C^1(\mathbb{R})$.

4) We prove that : $\forall x \in \mathbb{R},\ \ |\sin x| \leq |x|$.

Let $x \in \mathbb{R}$, then we have $x=0$ or $x>0$ or $x<0$.

- If $x=0:$ $|\sin 0| = 0 \leq |0| = 0$.

- If $x>0:$ we set $g(t)=\sin t$ on $[0,x]$,

$g$ is continuous on $[0,x]$ and derivable on $]0,x[$, then, according to the Mean Value Theorem, we have
$\exists c \in ]0,x[$ such that $g(x)-g(0)=(x-0)g'(c)$, then $\sin x = x\cos c$,

so $|\sin x| = |x||\cos c| \leq |x| \quad$ (since $|\cos c| \leq 1$), hence $|\sin x| \leq |x|$.

- If $x<0:$ we set $g(t)=\sin t$ on $[x,0]$

$g$ is continuous on $[x,0]$ and derivable on $]x,0[$, then, according to the Mean Value Theorem, we have
$\exists c \in ]x,0[$ such that $g(x)-g(0)=(x-0)g'(c)$, then $\sin x = x\cos c$,

so $|\sin x| = |x||\cos c| \leq |x| \quad$ (since $|\cos c| \leq 1$), hence $|\sin x| \leq |x|$.

Therefore $\forall x \in \mathbb{R},\ \ |\sin x| \leq |x|$.

**Exercise 210** *Let the function* $f(x) = \dfrac{x^2}{x+2}e^{\frac{-1}{x^2}}$.

1) *Find the domain of definition* $D_f$ *of the function* $f$.

2) *Study the continuity and derivability of the function* $f$.

3) *Study the extension by continuity of* $f$.

4) *Prove that* $\forall x \in \mathbb{R},\ \ e^x \geq x+1$.

**Solution :**

$f(x) = \dfrac{x^2}{x+2}e^{\frac{-1}{x^2}}$.

1) $D_f = \mathbb{R} \setminus \{-2,0\}$.

2) Study the continuity and derivability of the function $f$ :

$f$ is continuous and derivable on $D_f$ because it is the sum, the product, and the composition of continuous and derivable functions on $D_f$.

3) The extension by continuity of $f$ at $x_0 = -2 \notin D_f$,

$$\lim_{x \longrightarrow -2} f(x) = \lim_{x \longrightarrow -2} \frac{x^2}{x+2} e^{\frac{-1}{x^2}} = \pm\infty,$$

then $f$ does not admit an extension by continuity at $x_0 = -2$.

The extension by continuity of $f$ at $x_0 = 0 \notin D_f$,

$$\lim_{x \longrightarrow 0} f(x) = \lim_{x \longrightarrow 0} \frac{x^2}{x+2} e^{\frac{-1}{x^2}} = 0,$$

then $f$ admit an extension by continuity at 0 and its extension is :

$$\widetilde{f}(x) = \begin{cases} \dfrac{x^2}{x+2} e^{\frac{-1}{x^2}}, & x \in \mathbb{R} \setminus \{0, -2\}, \\ 0, & x = 0. \end{cases}$$

4) We prove $\forall x \in \mathbb{R},\ \ e^x \geq x + 1$.

We study three cases :

$1^{st}$**case** : if $x = 0$ :

we directly substitute into the inequality : $e^0 = 1 \geq 0 + 1$, so the inequality holds.

$2^{nd}$**case** : if $x > 0$ :

We apply the Mean Value Theorem, we take $g(t) = e^t$ and $[a, b] = [0, x]$,

$g$ is continuous on $[0, x]$ and derivable on $]0, x[$, then, according to the Mean Value Theorem

$\exists c \in ]0, x[$ such that $g(x) - g(0) = (x - 0)g'(c)$,

then $e^x - e^0 = (x - 0)e^c$, hence $e^x - 1 = xe^c > x$ (since $c > 0 \Longrightarrow e^c > 1$),

thus $e^x > x + 1$.

$3^{rd}$**case** : if $x < 0$ :

We apply the Mean Value Theorem, we take $g(t) = e^t$ and $[a, b] = [x, 0]$,

$g$ is continuous on $[x, 0]$ and derivable on $]x, 0[$, then, according to the Mean Value Theorem

$\exists c \in ]x, 0[$ such $g(x) - g(0) = (x - 0)g'(c)$,

then $e^x - e^0 = (x - 0)e^c$,

hence $e^x - 1 = xe^c > x$ (since $c < 0 \Longrightarrow e^c < 1$ and $x < 0$),

therefore $e^x > x + 1$.

**Conclusion** : $\forall x \in \mathbb{R},\ \ e^x \geq x + 1$.

**Exercise 211** *Let $f$ be a function definie by*

$$f(x)=\begin{cases} x^3(2-3\ln(x^2)) & , \quad x\neq 0,\\ 0 & , \quad x=0.\end{cases}$$

1) *Justify the application of Rolle's Theorem to the function $f$ on $\left[0,e^{\frac{1}{3}}\right]$.*

2) *Specify the value $c\in\left]0,e^{\frac{1}{3}}\right[ \ / \ f'(c)=0$.*

**Solution :**
1) We justify the application of Rolle's Theorem to the function $f$ on $\left[0,e^{\frac{1}{3}}\right]$.

We study the continuity of $f$ on $D_f=\mathbb{R}$ :

$f$ is continuous on $\mathbb{R}^*$ because it is the sum, the product, and the composition of continuous functions on $\mathbb{R}^*$.

Continuity of $f$ at $0$ : $f(0)=0$,

$\lim\limits_{x\longrightarrow 0} f(x)=\lim\limits_{x\longrightarrow 0} x^3(2-3\ln(x^2))=\lim\limits_{x\longrightarrow 0}(2x^3-3x^3\ln(x^2))=0=f(0)$,

then $f$ is continuous at 0, so $f$ is continuous on $\mathbb{R}$.

$f$ is derivable on $\mathbb{R}^*$ because it is the sum, the product, and the composition of derivable functions on $\mathbb{R}^*$.

Then $f$ is continuous on $\left[0,e^{\frac{1}{3}}\right]$ and it is derivable on $\left]0,e^{\frac{1}{3}}\right[$, moreover $f(0)=f(e^{\frac{1}{3}})=0$,

Thus, according to Rolle's Theorem, $\exists c\in\left]0,e^{\frac{1}{3}}\right[ \ / \ f'(c)=0$.

2) Let's specify the value $c\in\left]0,e^{\frac{1}{3}}\right[ \ / \ f'(c)=0$.

For $x\neq 0$, $f'(x)=3x^2(2-3\ln(x^2))+x^3\dfrac{(-6x)}{x^2}=-9x^2\ln(x^2)$,

$f'(c)=0\Longleftrightarrow -9c^2\ln(c^2)=0$,

since $c\in\left]0,e^{\frac{1}{3}}\right[$, so $c=1$.

**Exercise 212** *Let the function $f(x)=\dfrac{4e^x}{e^x+1}$.*

1) *Find the domain of definition $D_f$ de la fonction $f$.*

2) *Study the continuity and derivability of the function $f$ and calculate $f'$.*

3) *Prove that $f$ admit a unique fixed point in $]3,4[$.*

**Solution :**
1) $D_f = \mathbb{R}$.

2) Study of the continuity and the derivability of the function $f$ :

$f$ is continuous and derivable on $\mathbb{R}$ because it is the sum and the product of continuous and derivable functions on $\mathbb{R}$.

$\forall x \in \mathbb{R},\ \ f'(x) = \dfrac{4e^x}{(e^x+1)^2}.$

3) Let us show that $f$ admit a unique fixed point in $]3,4[$ :

the fixed point of $f$ is the solution of the equation $f(x) = x$.

We set $g(x) = f(x) - x$,

$g$ is continuous on $[3,4]$ because it is the sum of two continuous functions on $\mathbb{R}$, particularly on $[3,4]$,

moreover, we have $g(3) > 0$ and $g(4) < 0$,

thus, according to the Intermediate Value Theorem, $\exists c \in ]3,4[\ /\ g(c) = 0$.

Let us show that this point is unique: it is enough to show that the function $g$ is strictly monotone.

$g\prime(x) = \dfrac{4e^x}{(e^x+1)^2} - 1 = \dfrac{-(e^x-1)^2}{(e^x+1)^2} < 0,\ \ \forall x \in \mathbb{R}^*,$

then $g$ is strictly monotone, hence the point $c$ is unique.

We have $f(c) - c = 0$, d'où $f(c) = c$, so $f$ admit a unique fixed point.

# Chapter 5

# Circular functions and hyperbolic functions

## 5.1 Reciprocal circular functions

### 5.1.1 Arcsine function

Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1,1] \ / \ f(x) = \sin x$.

$f'(x) = \cos x$.

$f$ is continuous and strictly increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$f\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1,1]$.

Then, $f$ is a bijection and therefore $f$ has an inverse function $f^{-1}$.

$f^{-1} : [-1,1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \ / \ f^{-1}(x) = \arcsin x$.

$f^{-1}$ is continuous and strictly increasing on $[-1,1]$.

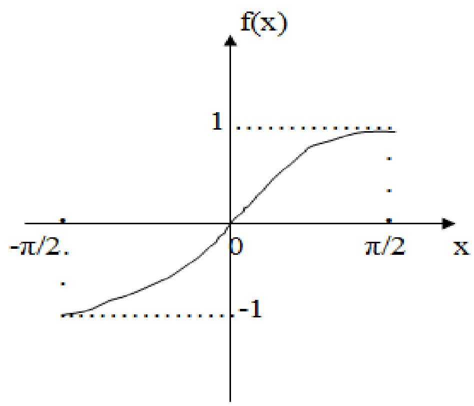
$\forall x \in [-1,1]\,,\ f^{-1}(x) = y \ / \ y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x = \sin y$.

**The derivative of the inverse function** $f^{-1}$:

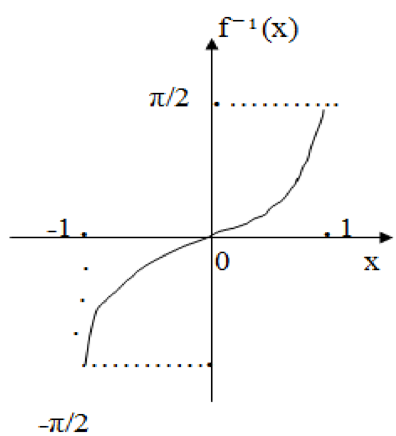
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \forall x \in \,]1,1[\,.$$

**Remark 213** *The graph of the inverse function is obtained by symmetry with respect to the first bisector* ($y = x$).

**Graphs**




Graph of $f(x) = \sin x$




Graph of $f^{-1}(x) = \arcsin x$

### 5.1.2 Arccosine function

Let $f : [0, \pi] \longrightarrow [-1, 1]$ / $f(x) = \cos x$.

$f'(x) = -\sin x$.

$f$ is continuous and strictly decreasing on $[0, \pi]$.

$f([0,\pi]) = [-1,1]$.

Then, $f$ is a bijection and therefore $f$ has an inverse function $f^{-1}$.

$f^{-1} : [-1,1] \longrightarrow [0,\pi] \ / \ f^{-1}(x) = \arccos x$.

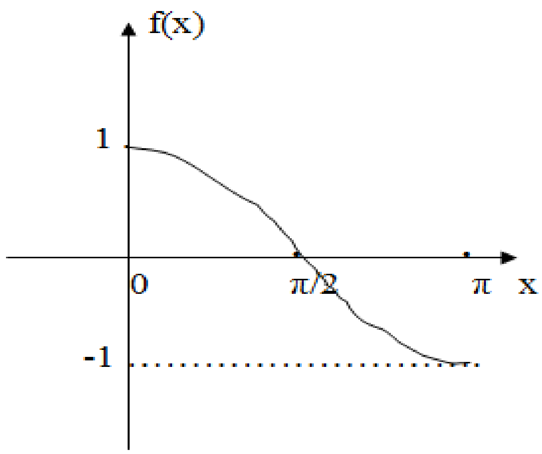
$f^{-1}$ is continuous and strictly decreasing on $[-1,1]$.

$\forall x \in [-1,1], \ f^{-1}(x) = y \ / \ y \in [0,\pi]$ et $x = \cos y$.

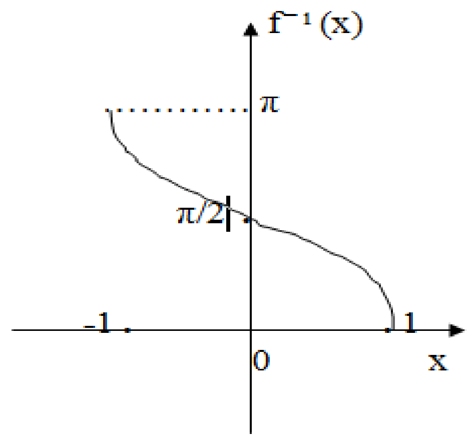
**The derivative of the inverse function $f^{-1}$:**

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, \forall x \in ]1,1[.$$

**Graphs**




Graph of $f(x) = \cos x$




Graph of $f^{-1}(x) = \arccos x$

**Property:**

$$\forall x \in [-1, 1], \arcsin x + \arccos x = \frac{\pi}{2}$$

Indeed, we set $f(x) = \arcsin x + \arccos x$,

$\forall x \in ]-1, 1[,\ \ f'(x) = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}} = 0,$

then, $f(x) = c :$ constant.

$c = f(0) = \arcsin 0 + \arccos 0 = 0 + \dfrac{\pi}{2} = \dfrac{\pi}{2},$

and we have $f(1) = f(-1) = \dfrac{\pi}{2}.$

Therefore, $\forall x \in [-1, 1],\ \ \arcsin x + \arccos x = \dfrac{\pi}{2}.$

### 5.1.3 Arctangent function

Let $f : \left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[ \longrightarrow \mathbb{R}\ \ /\ f(x) = \tan x = \dfrac{\sin x}{\cos x}.$

$f'(x) = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x.$

$f$ is continuous and strictly increasing on $\left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[.$

$f\left(\left]-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right[\right) = \mathbb{R}.$

Then, $f$ is a bijection and therefore $f$ has an inverse function $f^{-1}$.

$f^{-1} : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \ / \ f^{-1}(x) = \arctan x.$

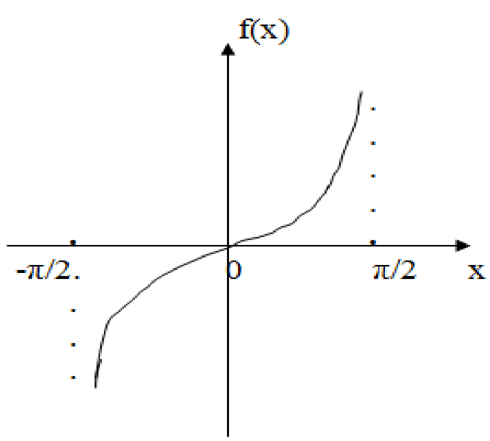
$f^{-1}$ is continuous and strictly increasing on $\mathbb{R}$.

$\forall x \in \mathbb{R}, f^{-1}(x) = y \ / \ y \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ et $x = \tan y$.

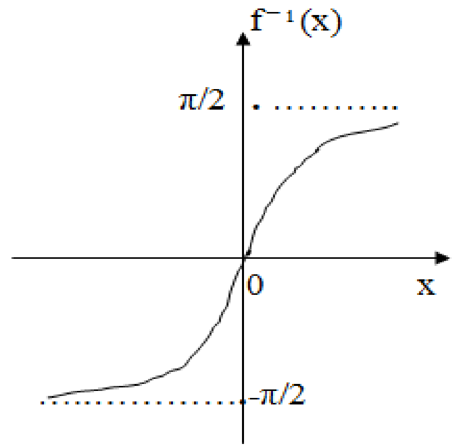
**The derivative of the inverse function $f^{-1}$:**

$$(\arctan x)' = \frac{1}{1+x^2}.$$

**Graphs:**




Graph of $f(x) = \tan x$




Graph of $f^{-1}(x) = \arctan x$

**Remark 214** *We introduce the function* arccot : $\mathbb{R} \longrightarrow ]0, \pi[$ *which is the inverse function of the restriction of the function* cot *on* $]0, \pi[$ $(\cot x = \dfrac{\cos x}{\sin x})$.

*We have :* $(\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$.

$\forall x \in \mathbb{R}, \arctan x + \operatorname{arccot} x = \dfrac{\pi}{2}$.

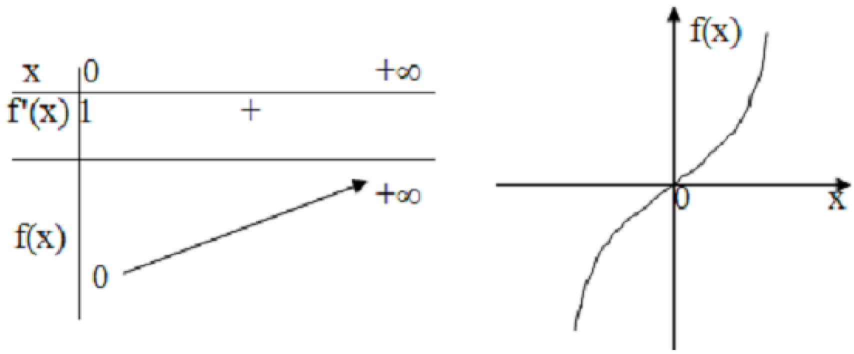
## 5.2 Hyperbolic functions and their inverses

### 5.2.1 Hyperbolic sine function and its inverse, Hyperbolic sine function argument"

Let $f : \mathbb{R} \longrightarrow \mathbb{R} \ / \ f(x) = shx = \dfrac{e^x - e^{-x}}{2}$.

$\forall x \in \mathbb{R},\ f'(x) = \dfrac{e^x + e^{-x}}{2} = chx > 0.$

**Remark 215** *$f$ is odd, so it is enough to study $f$ on $\mathbb{R}^+$, the rest of the graph* *can be deduced by symmetry with respect to the origin of the coordinate system.*




*Variations and graph of* $f(x) = shx$

$\lim\limits_{x \to +\infty} \dfrac{shx}{x} = +\infty,$

Then, there exists an asymptotic direction parallel to the axis $(OY)$.

$f$ is continuous and strictly increasing on $\mathbb{R}$.

$f(\mathbb{R}) = \mathbb{R}$.

Hence, $f$ is a bijection and therefore $f$ has an inverse function $f^{-1}$ :

$f^{-1} : \mathbb{R} \longrightarrow \mathbb{R} \ / \ f^{-1}(x) = \arg shx$.

$f^{-1}$ is continuous and strictly increasing on $\mathbb{R}$.

$\forall x \in \mathbb{R}$, $y = \arg shx \Longleftrightarrow y \in \mathbb{R}$ and $x = shy$.

**The derivative of the inverse function $f^{-1}$:**

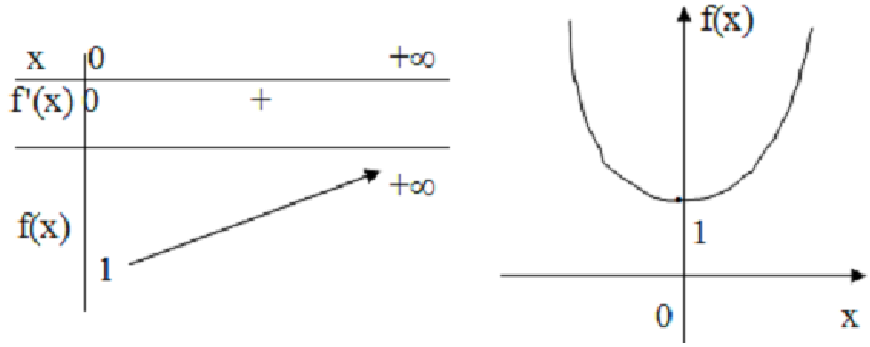
$$\forall x \in \mathbb{R}, (\arg shx)' = \frac{1}{\sqrt{1+x^2}}.$$

### 5.2.2 Hyperbolic cosine function and its inverse, Hyperbolic cosine function argument

Let $f : \mathbb{R} \longrightarrow \mathbb{R} \ / \ f(x) = chx = \dfrac{e^x + e^{-x}}{2}$.

$\forall x \in \left]0, +\infty\right[, f'(x) = \dfrac{e^x - e^{-x}}{2} = shx > 0.$

**Remark 216** *$f$ is an even function, so it is enough to study $f$ sur $\mathbb{R}_+$, and the rest of the graph can be deduced by symmetry with respect to the axis (Oy).*




*Variations and graph of $f(x) = chx$*

$\lim\limits_{x \to +\infty} \dfrac{chx}{x} = +\infty,$

then, there exists an asymptotic direction parallel to the (Oy) axis.

Since $f$ is not injective on $\mathbb{R}$, we consider its restriction on $[0, +\infty[$.

$f : [0, +\infty[ \longrightarrow [1, +\infty[ \quad / \quad f(x) = chx.$

$f$ is continuous and strictly increasing on $[0, +\infty[$.

$f([0, +\infty[) = [1, +\infty[$.

Then, $f$ is a bijection and consequently $f$ admits an inverse function $f^{-1}$ :

$f^{-1} : [1, +\infty[ \longrightarrow [0, +\infty[ \ / \ f^{-1}(x) = \arg chx.$

$f^{-1}$ is continuous and strictly increasing on $[1, +\infty[$.

$\forall x \geq 1, y = \arg chx \Longleftrightarrow y \geq 0$ and $x = chy.$

**The derivative of the inverse function $f^{-1}$:**

$$\forall x > 1, (\arg chx)' = \frac{1}{\sqrt{x^2 - 1}}.$$

**Properties :**

1) $chx + shx = e^x$.

2) $chx - shx = e^{-x}$.

3) $ch^2x - sh^2x = 1$.

4) $ch(x + y) = chx.chy + shx.shy$.

5) $ch(2x) = ch^2x + sh^2x = 1 + 2sh^2x = 2ch^2x - 1$.

6) $sh(x + y) = shx.chy + shy.chx$.

7) $sh(2x) = 2shx.chx$.

**Remark 217** *We can express the functions* $\arg shx$ *and* $\arg chx$ *using the logarithmic function :*

**1**) $x = shy \Longleftrightarrow y = \arg shx$,

$ch^2y - sh^2y = 1 \Longrightarrow ch^2y = 1 + sh^2y$ *and* $chy > 0$,

*then,* $chy = \sqrt{1 + sh^2y} = \sqrt{1 + x^2}$,

*hence,* $shy + chy = x + \sqrt{1 + x^2} = e^y$,

*therefore,* $y = \ln(x + \sqrt{1 + x^2})$,

*finally,* $\arg shx = \ln(x + \sqrt{1 + x^2})$.

**2**) $chy = x$ *and* $shy = \sqrt{x^2 - 1}$ *(since* $y \geq 0$*)*,

*then,* $chy + shy = x + \sqrt{x^2 - 1} = e^y$,

*hence,* $y = \ln(x + \sqrt{x^2 - 1})$,

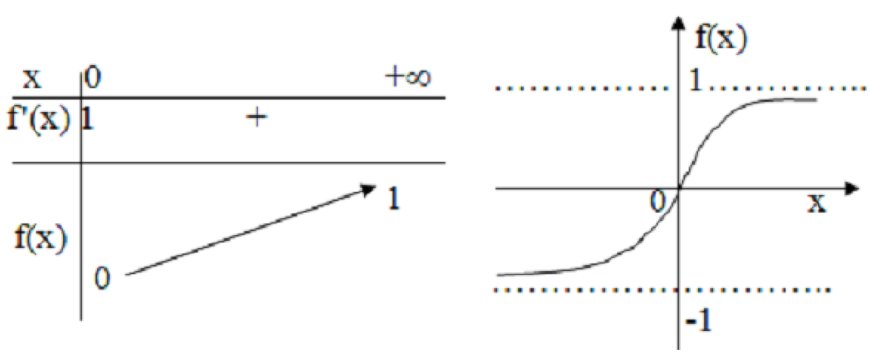
*therefore,* $\arg chx = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$.

### 5.2.3 Hyperbolic tangent function and its inverse hyperbolic tangent function argument

Let $f : \mathbb{R} \longrightarrow \mathbb{R} \ / \ f(x) = thx = \dfrac{shx}{chx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$.

$\forall x \in \mathbb{R}$, $f'(x) = \frac{1}{ch^2x} = 1 - th^2x > 0$.

**Remark 218** *$f$ is odd, so it is enough to study $f$ on $\mathbb{R}_+$, we can deduce the rest of the graph by symmetry with respect to the origin of the coordinate system*




*Variations and graph of $f(x) = thx$*

$f(\mathbb{R}) = ]-1,1[$.

Then, we consider the function $f : \mathbb{R} \longrightarrow ]-1,1[ \ / \ f(x) = thx$.

$f$ is continuous and strictly increasing on $\mathbb{R}$.

Then, $f$ is a bijection and consequently,$f$ has an inverse function $f^{-1}$.

$f^{-1} : ]-1,1[ \longrightarrow \mathbb{R} \ / \ f^{-1}(x) = \arg thx$.

$f^{-1}$ is continuous and strictly increasing on $]-1,1[$.

$\forall x \in ]-1,1[, \ \ y = \arg thx \Longleftrightarrow y \in \mathbb{R}$ and $x = thy$.

**The derivative of the inverse function $f^{-1}$:**

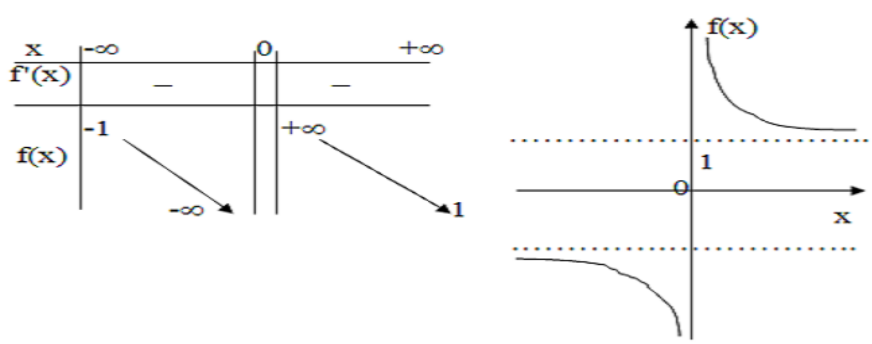
$$\forall x \in ]-1,1[, (\arg thx)' = \frac{1}{1-x^2}$$

### 5.2.4 Hyperbolic cotangent function and its inverse hyperbolic cotangent function argument

Let $f : \mathbb{R}^* \longrightarrow \mathbb{R} \ / \ f(x) = \coth x = \dfrac{1}{thx} = \dfrac{chx}{shx} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} = \dfrac{e^{2x}+1}{e^{2x}-1}$.

$\forall x \in \mathbb{R}^*, f'(x) = -\dfrac{1}{sh^2 x} = 1 - \coth^2 x < 0$.

**Remark 219** *$f$ is odd.*




*Variations and graph of $f(x) = \coth x$*

The lines with equations $x = 0$, $y = 1$ and $y = -1$ are asymptotes.

$f(\mathbb{R}^*) = \left]-\infty, -1\right[ \cup \left]1, +\infty\right[$.

Then, we consider the function $f : \mathbb{R}^* \longrightarrow \left]-\infty, -1\right[ \cup \left]1, +\infty\right[ \;\; / \;\; f(x) = \coth x$.

$f$ is continuous and strictement decreasing on $\mathbb{R}^*$.

then, $f$ is a bijection and therefore $f$ admits an inverse function $f^{-1}$.

$f^{-1} : \left]-\infty, -1\right[ \cup \left]1, +\infty\right[ \longrightarrow \mathbb{R}^* \;/\; f^{-1}(x) = \arg\coth x$.

$f^{-1}$ is continuous and strictly decreasing on$\left]-\infty, -1\right[ \cup \left]1, +\infty\right[$.

$\forall x \in \left]-\infty, -1\right[ \cup \left]1, +\infty\right[,\;\; y = \arg\coth x \Longleftrightarrow y \in \mathbb{R}^*$ and $x = \coth y$.

**The derivative of the inverse function** $f^{-1}$:

$$\forall x \in \left]-\infty, -1\right[ \cup \left]1, +\infty\right[, (\arg\coth x)' = \frac{1}{1 - x^2}.$$

## 5.3 Exercices

**Exercise 220** *Let the function* $f(x) = \arccos \dfrac{1 - x}{2}$.

1) *Find the domain of definition.* $D_f$ *of the function* $f$.

2) *Study the continuity and derivability of the function* $f$ *and calculate* $f'$.

**Solution :**
$f(x) = \arccos \dfrac{1 - x}{2}$.

1) The domain of definition $D_f$ of the function $f$ :

the function $\arccos x$ is defined on $[-1, 1]$, then we have

$D_f = \left\{ x \in \mathbb{R} \;/\; -1 \leq \dfrac{1 - x}{2} \leq 1 \right\}$,

$-1 \leq \dfrac{1 - x}{2} \leq 1 \Longleftrightarrow -2 \leq 1 - x \leq 2 \Longleftrightarrow -1 \leq x \leq 3$

hence, $D_f = [-1, 3]$.

2) Study of the continuity and derivability of the function $f$ .

• $f$ is continuous on $[-1, 3]$ because it is the sum, the product, and the composition of continuous functions on $[-1, 3]$.

• $f$ is derivable on $\left]-1, 3\right[$ because it is the sum, the product, and the composition of derivable functions on $\left]-1, 3\right[$.

$$\forall x \in ]-1,3[\ ,\ f'(x) = -\frac{1}{2}\left(\frac{-1}{\sqrt{1-(\frac{1-x}{2})^2}}\right) = \left(\frac{1}{\sqrt{-x^2+2x+3}}\right).$$

**Exercise 221** *Let the function* $f(x) = x - \arctan\frac{x+1}{x}$.

1) *Find the domain of definition* $D_f$ *of the function* $f$.

2) *Study the continuity and derivability of the function* $f$ *and calculate* $f'$.

3) *Study the extension by continuity of* $f$.

4) *Prove that :* $\forall x \geq 0, \dfrac{x}{1+x^2} \leq \arctan x \leq x$.

**Solution :**

1) $f(x) = x - \arctan\dfrac{x+1}{x}$.

The function $\arctan x$ is defined on $\mathbb{R}$, then $D_f = \mathbb{R}^*$.

2) Study of the continuity and derivability of the function $f$ :

$f$ is continuous and derivable on $\mathbb{R}^*$ because it is the sum, the product, and the composition of continuous and derivable functions on $\mathbb{R}^*$.

$$\forall x \in \mathbb{R}^*,\ f'(x) = 1 - \left(\frac{x+1}{x}\right)' \frac{1}{1+\left(\frac{x+1}{x}\right)^2} = 1 - \left(\frac{-1}{x^2}\right)\frac{1}{\frac{x^2+(x+1)^2}{x^2}},$$

$$f'(x) = 1 + \frac{1}{x^2+(x+1)^2}.$$

3) Study of the extension by continuity of $f$ at $x_0 = 0$ : $0 \notin D_f$

$$\lim_{x \overset{>}{\longrightarrow} 0} f(x) = \lim_{x \overset{>}{\longrightarrow} 0}\left(x - \arctan\frac{x+1}{x}\right) = -\frac{\pi}{2},$$

$$\lim_{x \overset{<}{\longrightarrow} 0} f(x) = \lim_{x \overset{<}{\longrightarrow} 0}\left(x - \arctan\frac{x+1}{x}\right) = \frac{\pi}{2},$$

then, $f$ does not admit a limit at 0.

Therefore, $f$ does not admit an extension by continuity at 0.

4) Let us show that : $\forall x \geq 0, \dfrac{x}{1+x^2} \leq \arctan x \leq x$.

We study two cases :

$1^{st}$**case** : if $x = 0$ :

we directly substitute into the inequality : $\frac{0}{1+0^2} \leq \arctan 0 = 0 \leq 0$,

so, the inequality holds.

$2^{nd}$**case** : if $x > 0$ :

we apply the Mean Value Theorem,

we take $g(t) = \arctan t$ and $[a, b] = [0, x]$,

$g'(t) = \frac{1}{1+t^2}$.

$g$ is continuous on $[0, x]$ and derivable on $]0, x[$, then, according to the Mean Value Theorem

$\exists c \in ]0, x[$ such that $g(x) - g(0) = (x - 0)g'(c)$,

then, $\arctan x - \arctan 0 = (x - 0)\frac{1}{1+c^2}$,

so, $\arctan x = \frac{x}{1+c^2}$.

On the other hand, we have

$$c \in ]0, x[ \Longrightarrow 0 < c < x \Longrightarrow 1 < 1 + c^2 < 1 + x^2,$$

then, $\frac{1}{1+x^2} < \frac{1}{1+c^2} < 1 \Longrightarrow \frac{x}{1+x^2} < \frac{x}{1+c^2} < x$,

thus, $\frac{x}{1+x^2} < \arctan x < x$.

**Conclusion :**
$\forall x \geq 0$, $\frac{x}{1+x^2} \leq \arctan x \leq x$.

**Exercise 222** *Let the function* $f(x) = \arcsin \frac{x}{1+x}$.

1) *Find the domain of definition* $D_f$ *of the function* $f$.

2) *Study the continuity and derivability of the function* $f$ *and calculate* $f'$.

**Solution :**
1) $f(x) = \arcsin \frac{x}{1+x}$.

The function $\arcsin x$ is defined on $[-1, 1]$, then

$D_f = \left\{x \in \mathbb{R} / -1 \leq \frac{x}{1+x} \leq 1\right\} = \left\{x \in \mathbb{R} / \left|\frac{x}{1+x}\right| \leq 1\right\}$,

$\left|\frac{x}{1+x}\right| \leq 1 \Longleftrightarrow \frac{x^2}{(1+x)^2} \leq 1 \Longleftrightarrow x^2 \leq (1+x)^2 = x^2 + 2x + 1$

$$\iff 2x+1 \geq 0 \iff x \geq -\frac{1}{2}.$$

Then, $D_f = \left[-\frac{1}{2}, +\infty\right[$.

2) Study of the continuity and derivability of the function $f$.

$f$ is continuous on $D_f$ because it is the sum, the product, and the composition of continuous functions on $D_f$.

$f$ is derivable on $\left]-\frac{1}{2}, +\infty\right[$ because it is the sum, the product, and the composition of derivable functions on $\left]-\frac{1}{2}, +\infty\right[$.

$\forall x \in \left]-\frac{1}{2}, +\infty\right[$,

$$f'(x) = \left(\frac{x}{1+x}\right)' \frac{1}{\sqrt{1-\left(\frac{x}{1+x}\right)^2}} = \frac{1}{(1+x)^2} \frac{1}{\sqrt{\frac{(x+1)^2 - x^2}{(1+x)^2}}},$$

$\forall x \in \left]-\frac{1}{2}, +\infty\right[$, $f'(x) = \dfrac{1}{(1+x)} \dfrac{1}{\sqrt{2x+1}}$.

# Chapter 6

# Usual formulas

## 6.1 Partial sum of an arithmetic sequence

$U_n = U_0 + nr, \;\; r \in \mathbb{R}^*.$

$$S_n = U_0 + U_1 + U_2 + ..... + U_n = (U_0 + U_n)\frac{n+1}{2}.$$

## 6.2 Partial sum of a geometric sequence

$U_n = U_0 q^n, \;\; q \neq 1,$

$$S_n = U_0 + U_1 + U_2 + ..... + U_n = U_0 \left(\frac{1-q^{n+1}}{1-q}\right).$$

If $q = 1, S_n = (n+1)U_0$.

$\lim\limits_{n \longrightarrow +\infty} q^n = 0 \Longleftrightarrow -1 < q < 1$

## 6.3 Trigonometry Formulas

1) $\sin(a+b) = \sin a \cos b + \sin b \cos a$, so $\sin 2a = 2 \sin a \cos a$.

2) $\sin(a-b) = \sin a \cos b - \sin b \cos a$.

3) $\cos(a+b) = \cos a \cos b - \sin a \sin b$, so $\cos 2a = \cos^2 a - \sin^2 a$.

4) $\cos(a-b) = \cos a \cos b + \sin a \sin b$.

5) $\cos 2a = 2\cos^2 a - 1$, so $\cos^2 a = \dfrac{\cos 2a + 1}{2}$.

6) $\cos 2a = 1 - 2\sin^2 a$, so $\sin^2 a = \dfrac{1 - \cos 2a}{2}$.

7) $\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$.

8) $\sin p - \sin q = 2\sin\frac{p-q}{2}\cos\frac{p+q}{2}$.

9) $\cos p + \cos q = 2\cos\frac{p-q}{2}\cos\frac{p+q}{2}$.

10) $\cos p - \cos q = -2\sin\frac{p-q}{2}\sin\frac{p+q}{2}$.

11) $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a.\tan b}$.

12) $\tan(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a.\tan b}$.

**Relation between sine and cosine**
$\sin^2 x + \cos^2 x = 1, \quad \forall x \in \mathbb{R}$.

## 6.4 Common values

| | | | | | | |
|---|---|---|---|---|---|---|
| nombre | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ |
| sinus | $0$ | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ |
| cosinus | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | $0$ | $-1$ |
| tangente | $0$ | $\frac{\sqrt{3}}{3}$ | $1$ | $\sqrt{3}$ | | $0$ |

## 6.5 Properties of hyperbolic functions

Hyperbolic sine : $shx = \dfrac{e^x - e^{-x}}{2}, \ \forall x \in \mathbb{R}$.

Hyperbolic cosine : $chx = \dfrac{e^x + e^{-x}}{2}, \ \forall x \in \mathbb{R}$.

1) $chx + shx = e^x$.

2) $chx - shx = e^{-x}$.

3) $ch^2x - sh^2x = 1$.

4) $ch(x+y) = chx.chy + shx.shy$.

5) $ch(2x) = ch^2x + sh^2x = 1 + 2sh^2x = 2ch^2x - 1$.

6) $sh(x+y) = shx.chy + shy.chx$.

7) $sh(2x) = 2shx.chx$.

## 6.6 Derivatives of usual functions

| The function | The derivative |
|---|---|
| $f(x) = x^n$ | $f'(x) = nx^{n-1}, \forall x \in \mathbb{R}$ |
| $f(x) = \ln x$ | $f'(x) = \dfrac{1}{x}, \forall x > 0$ |
| $f(x) = e^x$ | $f'(x) = e^x, \forall x \in \mathbb{R}$ |
| $f(x) = \sqrt{x}$ | $f'(x) = \dfrac{1}{2\sqrt{x}}, \forall x > 0$ |
| $f(x) = \sin x$ | $f'(x) = \cos x, \forall x \in \mathbb{R}$ |
| $f(x) = \cos x$ | $f'(x) = -\sin x, \forall x \in \mathbb{R}$ |
| $f(x) = \tan x = \dfrac{\sin x}{\cos x}$ | $f'(x) = \dfrac{1}{\cos^2 x} = 1 + \tan^2 x, x \neq \frac{\pi}{2} + k\pi$ |
| $f(x) = shx = \dfrac{e^x - e^{-x}}{2}$ | $f'(x) = chx = \dfrac{e^x + e^{-x}}{2}, \forall x \in \mathbb{R}$ |
| $f(x) = chx$ | $f'(x) = shx, \forall x \in \mathbb{R}$ |
| $f(x) = thx = \dfrac{shx}{chx}$ | $f'(x) = \dfrac{1}{ch^2x} = 1 - th^2x, \forall x \in \mathbb{R}$ |
| $f(x) = \arcsin x, \forall x \in [-1, 1]$ | $f'(x) = \dfrac{1}{\sqrt{1-x^2}}, \forall x \in ]-1, 1[$ |
| $f(x) = \arccos x, \forall x \in [-1, 1]$ | $f'(x) = \dfrac{-1}{\sqrt{1-x^2}}, \forall x \in ]-1, 1[$ |
| $f(x) = \arctan x$ | $f'(x) = \dfrac{1}{1+x^2}, \forall x \in \mathbb{R}$ |
| $f(x) = \arg shx$ | $f'(x) = \dfrac{1}{\sqrt{x^2+1}}$ |
| $f(x) = \arg chx$ | $f'(x) = \dfrac{1}{\sqrt{x^2-1}}$ |
| $f(x) = \arg thx$ | $f'(x) = \dfrac{1}{1-x^2}$ |
| $f(x) = (U(x))^n$ | $f'(x) = nU'(x)U^{n-1}(x)$ |
| $f(x) = \ln(U(x))$ | $f'(x) = \frac{U'(x)}{U(x)}$ |
| $f(x) = e^{ax}$ | $f'(x) = ae^{ax}, \forall x \in \mathbb{R}$ |

## 6.7 Lexicon

**A**

- Absolute value : valeur absolue.
- Absolute convergence : convergence absolue.
- Almost : presque.
- Analysis : analyse.
- Antisymetric : antisymétrique.
- Apex : sommet.
- Argument : argument.
- Arithmetic : arithmétique.
- Array : tableau.
- Assume : supposer.
- Assumption : supposition.
- Axiom : axiome.
- Axis : axe.

**B**

- Basis : base.
- Bijective : bijective.
- Bounded : borné.
- Bracket : parenthèse.
- By induction : par récurrence.

**C**

- Calculus : calcul.
- Cartesian coordinate system.: Repère cartésien.
- Cauchy sequence : suite de Cauchy.
- Center : centre
- Characteristic : caractéristique.
- Characteristic polynomial : polynôme caractéristique.
- Circle : cercle.
- Closed : fermé.
- Coefficient : coefficient.
- Combination : combinaison.
- Common factor : facteur commun.
- Commutative : commutatif.
- Complete : complet.
- Complex number : nombre complexe.
- Computation : calcul.
- Consequently : par conséquent.
- Constant : constante.
- Continuity : continuité.
- Continuous (function) : continue (fonction).
- Contraction : contraction.
- Convergence : convergence.
- Converge to a limit : converger vers une limite.
- Converse of a theorem : réciproque d'un théorème.

- Conversely : réciproquement.
- Coordinate : coordonnée.
- Cosine : cosinus.
- Countable : dénombrable.
- Counterexample : contre-exemple.
- Coverage of a set : recouvrement d'un ensemble.
- Cube root : racine cubique.
- Curve : courbe.

**D**

- Decomposition : décomposition.
- Decreasing function : fonction décroissante.
- Defined : défini.
- Degree : degré.
- Delete (to) : supprimer.
- Denote : noter.
- Density : densité.
- Derivative : dérivée.
- Direct sum : somme directe.
- Divide : diviser.
- Dot : point.

**E**

- Eigenvalue : valeur propre.
- Eigenvector : vecteur propre.
- Element : élément.
- Endpoint : Extrémité.
- Entire function : fonction entière.
- Equality : égalité.
- Equation : équation.
- Equilateral triangle : triangle equilatéral.
- Equivalence relation : relation d'équivalence.
- Equivalent : équivalent
- Euclidean : euclidien.
- Even : pair.
- Everywhere : partout.
- Exact : exact.
- Example : exemple.
- Exponential : exponentiel.

**F**

- Factorial : factoriel.
- Factorise : factoriser
- Field : corps.
- Finite : fini.
- Finite dimensional real vector space : espace vectoriel réel de dimension finie
- Fixed : fixe.
- Fixed point : point fixe.

- Formula : formule.
- Fractional line : trait de fraction.
- Free : libre.
- Function : fonction.
- Fundamental : fondamental.

**G**

- Graph : graphe.
- Greatest : plus grand (le).
- Greatest common divisor (gcd) : pgcd.
- Group : groupe.

**H**

- Higher derivative : dérivée d'ordre supérieur.
- Homogeneous : homogène.
- However : toutefois.
- Hyperbola : hyperbole.
- Hypotenuse : hypoténuse.
- Hypothesis : hypothèse.

**I**

- Identity : identité.
- Identity element : élément neutre.
- If and only if : si et seulement si.
- Increasing function : fonction croissante.
- Indeed : en effet.
- Independent : indépendant.
- Induction : récurrence.
- Inequality : inégalité.
- Infimum (greatest lower bound) : borne inférieure.
- Infinite : infini.
- Integer number : nombre entier.
- Integral : intégrale.
- Intermediate value theorem : théorème des valeurs intermédiaires.
- Interval : intervalle.
- inverse image : image réciproque.
- Invertible : inversible.
- Involve : impliquer.
- Irreducible : irréductible.
- Isocel triangle : triangle isocèle
- Isolated : isolé.
- Isomorphism : isomorphisme.

**J**

**K**

- Kernel : noyau.

**L**

- Law of composition : loi de composition.
- Least : plus petit.
- Least common multiple (lcm) : ppcm.

- Lemma : lemme.
- Length : longueur.
- Less than : plus petit que
- Let.....be : soit.
- Limit : limite
- Linear : linéaire.
- Linearly independent family : famille libre.
- Lower limit : limite inférieure.
- Lower bound : minorant.

**M**

- Major : majeur.
- Majorized : majoré
- Manifold : variété.
- Map : application.
- Maximal : maximal.
- Mean : moyenne.
- Meet of two sets : intersection de deux ensembles.
- Merely : seulement.
- Minimal : minimal.
- Minorized : minoré.
- Monic : unitaire.
- Monotonic function : fonction monotone.
- Multiplicity : multiplicité.
- Multiply : multiplier.

**N**

- Necessary condition : condition nécessaire.
- Negligible : négligeable.
- Neighborhood : voisinage.
- Neperian logarithm : logarithme népérien.
- Non-empty : non vide.
- Not all zero : non tous nuls.
- Null : nul.
- Number : nombre.
- Numerator : numérateur.

**O**

- Object : objet.
- Odd : impair.
- One-to-one map : application injective.
- Onto (a map) : surjective.
- Open : ouvert.
- Operator : opérateur.
- Order : ordre.
- Order or multiplicity of a root : ordre de multiplicité d'une racine.
- Order relation : relation d'ordre.
- Ordinate : ordonnée.

**P**

- Parameter : paramètre
- Partial fraction expansion : décomposition en éléments simples.
- Partial order : relation d'ordre.
- Partition : partition.
- Perfect : parfait.
- Period : période.
- Periodicity : périodicité.
- Permutation : permutation.
- Plane : plan.
- Point : point.
- Polynomial : polynôme.
- Power : puissance.
- Prime : premier.
- Prime number : nombre premier.
- Product : produit.
- Proof : preuve.
- Proper : propre.
- Property : propriété.
- Pythagorean triple : triplet pythagoricien.

**Q**

**R**

- Radius : rayon
- Raise to the power n : élever à la puissance n.
- Range : image.
- Rank : rang.
- Ratio : rapport.
- Rational function : fonction rationnelle.
- Real number : nombre réel.
- Rectangle : rectangle.
- Reduced : réduit.
- Regular : régulier
- Relatively prime integers : entiers premiers entre eux.
- Remark : remarque.
- representation : représentation.
- Right-hand side : membre de droite.
- Ring : anneau.
- Root : racine.
- Row : ligne.
- Rule : règle.
- Ruler : règle (instrument).

**S**

- Scalar : scalaire.
- Schwarz inequality : inégalité de Schwarz.
- Section : section.
- Segment : segment.
- Sequence : suite.

- Series : série.
- Set : ensemble.
- Several : plusieurs.
- Shape : forme.
- Sign : signe.
- Sine : sinus.
- Singular : singulier.
- Size : taille.
- Small : petit.
- Smooth : lisse.
- Space : espace.
- Square : élever au carré.
- Square : carré.
- Square root : racine carré.
- Star : Etoile.
- Strictly : strictement
- Sub : sous-
- Subgroup : sous-groupe.
- Subset : sous-ensemble (partie).
- Subspace : sous-espace.
- Subtract : soustraire.
- Subtraction : soustraction.
- Sufficient : suffisant.
- Sufficient condition : condition suffisante.
- Sum : somme.
- Summarize (to) : résumer.
- Support : support.
- Supremum (least upper bound) : borne supérieure.
- Surface : surface.
- Symmetric : symétrique.
- Symmetry : symétrie.
- System of linear equations : système d'équations linéaires.

**T**
- Tangent : tangente.
- Term : terme.
- Theorem : théorème.
- Theory : théorie.
- Totally ordered set : ensemble totalement ordonné.
- Trace : trace.
- Trajectory : trajectoire.
- Transform : transformation.
- Transitive : transitif.
- Translation : translation.
- Transpose : transposé.
- Trapezoid : trapèze.
- Triangle : triangle.

- Triangle inequality : inégalité triangulaire.
- Trivial : trivial.
- Type : type.

**U**

- Uncountable : indénombrable.
- Uniform continuity : continuité uniforme.
- Union : réunion.
- Universal : universel.
- Unknown : inconnue.
- Upper bound : majorant.

**V**

- Value : Valeur.
- Variable : variable.
- Vector : vecteur.
- Vector space : espace vectoriel.
- Volume : volume.

**W**

- Well-defined : bien défini.
- Width : largeur.
- Without loss of generality : sans perte de généralité.

**X**

**Y**

**Z**

- Zéro : zero.
- Zero of a polynomial : racine d'un polynôme.

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