ANALYSIS 2
Course & Exercises

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Contents

Chapter 1

Riemann integrals and Antiderivatives

1.1 Riemann integral

1.1.1 Subdivision

Definition 1 *Let $[a,b]$ be a closed bounded interval of $\mathbb{R}$. We call subdivision of $[a,b]$, any increasing sequence $d=(x_0,x_1,x_2,.....,x_n)$ of points of $[a,b]$ such that $x_0=a<x_1<x_2<.....<x_n=b$.*

We obtain n intervals $[x_i,x_{i+1}]$ $\quad (i\in\{0,1,2,....,(n-1)\})$, called partial intervals of the subdivision.

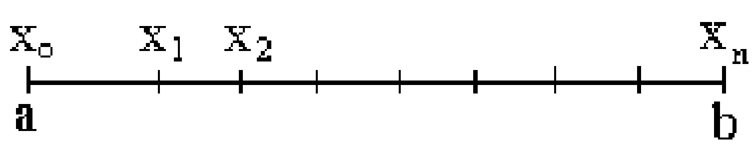
1.1.2 Darboux Sum

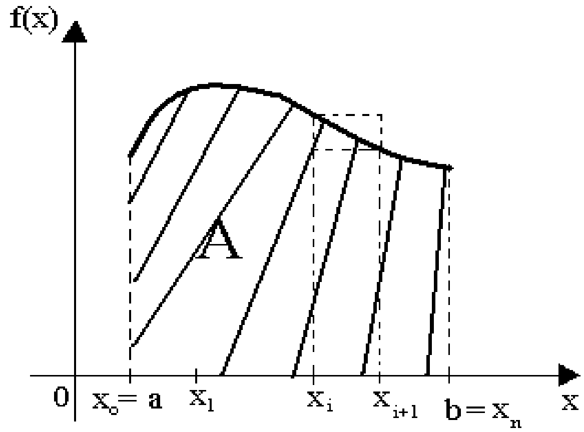
Let $f:[a,b]\longrightarrow\mathbb{R}$ be a bounded function, i.e.

$$\exists m,M\in\mathbb{R},\forall x\in[a,b]\,,\ m\leq f(x)\leq M.$$

Definition 2 *The integral of a positive function over the interval $[a,b]$ is the area of the region A enclosed by the curve of f, the axis (OX) and the two lines with equations $x=a$ and $x=b$.*

We consider the subdivision $d=(x_0,x_1,x_2,.....,x_n)$ of the interval $[a,b]$.





We set

$$m_i = \inf_{x\in[x_i,x_{i+1}]} f(x), \qquad i \in \{0, 1, 2, ..., (n-1)\},$$

$$M_i = \sup_{x\in[x_i,x_{i+1}]} f(x), \qquad i \in \{0, 1, 2, ..., (n-1)\}.$$

Definition 3 *- The lower Darboux sum is the following surface :*

$$s(f,d) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i).$$

-The upper Darboux sum is the following surface :

$$S(f,d) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i).$$

Remark 4 *Since* $m_i \leq M_i$*, then we have* $s(f,d) \leq A \leq S(f,d)$.

1.1.3 Lower Darboux integral and upper Darboux integral

Definition 5 *We define the following two sets :*
$D_s(f) = \{s(f,d)\ /\ d$ *subdivision of* $[a,b]\}$,
$D_S(f) = \{S(f,d)\ /\ d$ *subdivision of* $[a,b]\}$.

- The lower Darboux integral of f *over* $[a,b]$ *is the following value :*

$$\inf \int_a^b f(x)dx := \sup D_s(f).$$

- *The upper Darboux integral of f over $[a, b]$ s the following value :*

$$\sup \int_a^b f(x)dx := \inf D_S(f).$$

Remark 6 *We have $s(f, d) \leq \sup D_s(f) \leq \inf D_S(f) \leq S(f, d)$, therefore*

$$s(f, d) \leq \inf \int_a^b f(x)dx \leq \sup \int_a^b f(x)dx \leq S(f, d).$$

1.1.4 Riemann integral

Definition 7 *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. We say that f is Riemann-integrable on $[a, b]$ if*

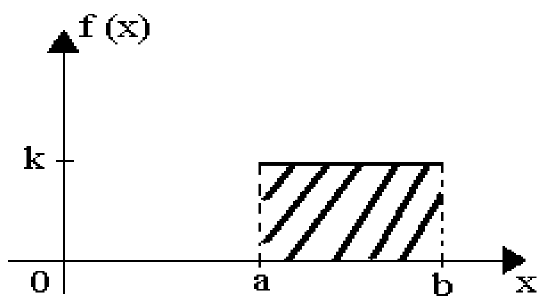
$$\inf \int_a^b f(x)dx = \sup \int_a^b f(x)dx.$$

Remark 8 *The common value of the lower and upper Darboux integrals is then called the Riemann integral of f over $[a, b]$ and it is denoted*

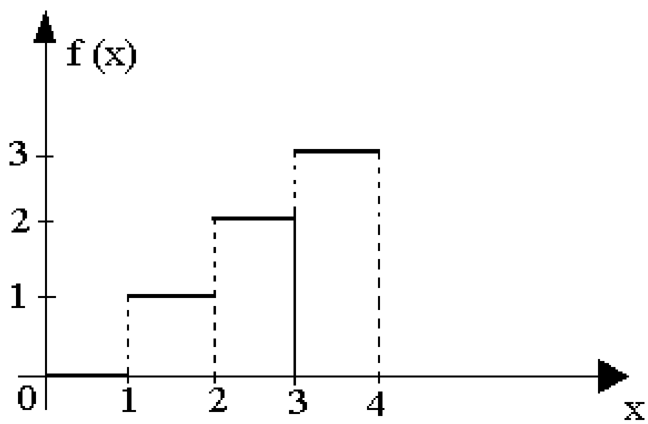
$$\int_a^b f(x)dx = \inf \int_a^b f(x)dx = \sup \int_a^b f(x)dx.$$

Example 9 $f : [a, b] \longrightarrow \mathbb{R} \ / \ f(x) = k, \quad k \in \mathbb{R}.$

$\int_a^b f(x)dx = k(b - a).$



Example 10 $\int_0^4 [x]\, dx = 0 + 1(2 - 1) + 2(3 - 2) + 3(4 - 3) = 6.$



1.1.5 Riemann sum

Definition 11 *Let* $c_i \in [x_i, x_{i+1}]$. *The sum* $\sigma(f,d) = \sum\limits_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i)$ *is called the Riemann sum of f corresponding to d and* $C = (c_0, \ldots, c_{n-1})$.

Remark 12 *Since* $x_i \leq c_i \leq x_{i+1}$, *then we have* $m_i \leq f(c_i) \leq M_i$ *and hence we obtain* $s(f,d) \leq \sigma(f,d) \leq S(f,d)$.

The step size of the subdivision

Let the subdivision $d = (x_0, x_1, \ldots, x_n)$ of the interval $[a,b]$. The real number $\delta(d) = \max\limits_{0 \leq i \leq n-1} (x_{i+1} - x_i)$ is called the step size of the subdivision d of the interval $[a,b]$.

Theorem 13 *If f is Riemann integrable over* $[a,b]$, *then*

$$\lim_{\delta(d) \longrightarrow 0} \sigma(f,d) = \int_a^b f(x)dx.$$

Theorem 14 *Any function continuous on* $[a,b]$ *is integrable over* $[a,b]$.

Consequence :

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function on $[a,b]$, then f is integrable sur $[a,b]$. We consider the following uniform subdivision $((x_{i+1} - x_i) = \text{constant})$:

$$d_n = (x_0 = a,\ x_1 = a + \frac{b-a}{n},\ x_2 = a + 2\frac{b-a}{n}, ..., x_i = a + i\frac{b-a}{n}, ..., x_n = b).$$

It is an arithmetic sequence with a common difference $r = \dfrac{b-a}{n} = \delta(d_n) = x_{i+1} - x_i$. We take $c_i = x_i = a + i\dfrac{b-a}{n}$.

$$\begin{aligned} \sigma(f, d_n) &= \sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i) = \sum_{i=0}^{n-1} f(x_i)\left(\frac{b-a}{n}\right) \\ &= \left(\frac{b-a}{n}\right)\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n}), \end{aligned}$$

then

$$\lim_{n \longrightarrow +\infty} \sigma(f, d_n) = \lim_{\delta(d_n) \longrightarrow 0} \sigma(f, d_n) = \int_a^b f(x)dx,$$

hence

$$\lim_{n \longrightarrow +\infty} \left(\frac{b-a}{n}\right)\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n}) = \int_a^b f(x)dx.$$

Conclusion :

$$f \text{ continuous on } [a,b] \Longrightarrow \int_a^b f(x)dx = \lim_{n \longrightarrow +\infty} \left(\frac{b-a}{n}\right)\sum_{i=0}^{n-1} f(a + i\frac{b-a}{n}).$$

Special case : if $a = 0$ and $b = 1$, then

$$\int_0^1 f(x)dx = \lim_{n \longrightarrow +\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(\frac{i}{n}).$$

Example 15 *Using the definition, calculate the following integral :*

$$\begin{aligned} \int_a^b kx\ dx &= \lim_{n \longrightarrow +\infty} \left(\frac{b-a}{n}\right)\sum_{i=0}^{n-1} k(a + i\frac{b-a}{n}) \\ &= k(b-a)\lim_{n \longrightarrow +\infty} \frac{1}{n}\sum_{i=0}^{n-1} (a + i\frac{b-a}{n}) \\ &= k(b-a)\lim_{n \longrightarrow +\infty} \frac{1}{n}(\frac{n}{2})\left(a + a + (\frac{n-1}{n})(b-a)\right) \\ &= k(b-a)(\frac{b+a}{2}) = \frac{k}{2}(b^2 - a^2). \end{aligned}$$

Theorem 16 *Any function $f : [a,b] \longrightarrow \mathbb{R}$ monotonous is integrable on $[a,b]$.*

Theorem 17 *If a bounded function $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$ except at a finite number of points of $[a,b]$, then f is integrable on $[a,b]$.*

Example 18 $\int\limits_0^4 [x]\,dx = 0 + 1(2-1) + 2(3-2) + 3(4-3) = 6.$

The floor function $[x]$ *is not continuous at the points :* $1, 2, 3 \in [0,4]$.

1.1.6 Properties of the integral

Property 1 :

- If $a < b$, then $\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)dx.$

- If $a = b$, then $\int\limits_a^b f(x)dx = 0.$

Property 2 :

If f is an integrable function on $[a,b]$ and if $\forall x \in [a,b], f(x) \geq 0$, then $\int\limits_a^b f(x)dx \geq 0.$

Property 3 :

If f and g are integrable functions on $[a,b]$, then the function $(f+g)$ is integrable on $[a,b]$ and we have $\int\limits_a^b (f+g)(x)dx = \int\limits_a^b f(x)dx + \int\limits_a^b g(x)dx.$

Property 4 :

If f is an integrable function on $[a,b]$, then the function λf $(\lambda \in \mathbb{R})$ is integrable on $[a,b]$ and we have $\int\limits_a^b \lambda f(x)dx = \lambda \int\limits_a^b f(x)dx.$

Remark 19 *From propositions* 3 *and* 4*, it follows that the set of functions integrable over* $[a,b]$ *is a vector space on* $\mathbb{R}$ *denoted* $R[a,b]$.

Property 5 :

If f and g are integrable functions on $[a,b]$ and if $\forall x \in [a,b]$, $f(x) \geq g(x)$, then $\int\limits_a^b f(x)dx \geq \int\limits_a^b g(x)dx.$

Property 6 :

If f is an integrable function on $[a,b]$, then f is integrable over each interval $[\alpha,\beta] \subset [a,b]$.

Property 7 :

1) Let $c \in]a,b[$. If f is integrable separately over $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$.

2) If f is integrable on $[a,b]$ and $c \in]a,b[$, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Property 8 :

If f is an integrable function on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and we have $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\, dx$.

Property 9 :

If f and g are integrable functions on $[a,b]$, then the function $(f.g)$ is integrable on $[a,b]$.

Theorem 20 ***(Schwarz inequality)***

Let f and g be two integrable functions on $[a,b]$, then

$$\left(\int_a^b f(x)g(x)dx \right)^2 \leq \int_a^b f^2(x)dx. \int_a^b g^2(x)dx.$$

Theorem 21 ***(Mean Value Formula)***

Let f and g be two integrable functions on $[a,b]$, g having a constant sign on $[a,b]$ ($g \geq 0$ or $g \leq 0$). We set $M = \sup_{x \in [a,b]} f(x)$ and $m = \inf_{x \in [a,b]} f(x)$.

Then, there exists $\mu \in [m, M]$ / $\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx$.

If moreover f is continuous, there exists $c \in [a,b]$ such that $\mu = f(c)$,

i.e. $\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx$.

Example 22 *Using the mean value formula, calculate the following limit :* $\lim_{x \longrightarrow 0} \int_x^{kx} \frac{\cos t}{t} dt, \quad k > 0.$

We set $f(t) = \cos t$ *and* $g(t) = \frac{1}{t}$. *The functions* f *and* g *are continuous on* $[x, kx]$ $(x \neq 0)$, *then* f *and* g *are integrable on* $[x, kx]$. *The function* g *has a constant sign on* $[x, kx]$, *then from the mean value formula, there exists* $c \in [x, kx]$ *such that*

$$\begin{aligned}\lim_{x \longrightarrow 0} \int_{x}^{kx} \frac{\cos t}{t} dt &= \lim_{x \longrightarrow 0} \cos c \int_{x}^{kx} \frac{1}{t} dt = \lim_{x \longrightarrow 0} \cos c. [\ln |t|]_{x}^{kx} = \lim_{x \longrightarrow 0} \cos c. \ln \left| \frac{kx}{x} \right| \\ &= \lim_{c \longrightarrow 0} \cos c. \ln |k| = \ln k.\end{aligned}$$

1.2 Integrals and antiderivatives

Let $f : [a, b] \longrightarrow \mathbb{R}$ be an integrable function on $[a, b]$ and $c \in [a, b]$ be a fixed point. We consider the function $F : [a, b] \longrightarrow \mathbb{R}$ such that $F(x) = \int_{c}^{x} f(t) dt.$

Theorem 23 1) *The function* F *is uniformly continuous on* $[a, b]$.

2) *If* f *is continuous on* $[a, b]$, *then* F *is derivable on* $[a, b]$ *and* $\forall x \in [a, b]$, $F'(x) = f(x)$.

1.2.1 Antiderivatives

Definition 24 *Let a function* $f : [a, b] \longrightarrow \mathbb{R}$. *We say that a derivable function* $F : [a, b] \longrightarrow \mathbb{R}$ *is an antiderivative of* f *if* $\forall x \in [a, b]$, $F'(x) = f(x)$.

Proposition 25 *Let* F_1 *and* F_2 *two antiderivatives of* f. *Then* $(F_1 - F_2)$ *is constant.*

Indeed, $(F_1 - F_2)' = F_1' - F_2' = f - f = 0 \Longrightarrow F_1 - F_2 = k, \quad (k \in \mathbb{R}).$

Conclusion :

If F is an antiderivaitive of f, then F is not unique because for all $k \in \mathbb{R}$, $F + k$ is also an antiderivative of f.

Theorem 26 *Every continuous function* $f : [a, b] \longrightarrow \mathbb{R}$, *has an antiderivative.*

The function $F : [a, b] \longrightarrow \mathbb{R}$ *such that* $F(x) = \int_{c}^{x} f(t) dt$, ($c \in [a, b]$ *a fixed point*) *is an antiderivative of* f.

Theorem 27 *Let f be a continuous function on $[a,b]$ and G any antiderivative of f. Then,* $\int\limits_a^b f(x)dx = G(b) - G(a)$.

Remark 28 1) *In the definition of the antiderivative, we can take instead of $[a,b]$, any interval I of $\mathbb{R}$, in particular $I = \mathbb{R}$.*

2) *If F is an antiderivative of f on $[a,b]$, this does not imply that f is continuous on $[a,b]$.*

Example 29 *Let the function defined by*

$$F(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

$$F'(0) = \lim_{x \longrightarrow 0} \frac{F(x) - F(0)}{x - 0} = \lim_{x \longrightarrow 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \longrightarrow 0} x \sin \frac{1}{x} = 0.$$

Then, F is derivable on $\mathbb{R}$ and its derivative is the following function

$$F'(x) = f(x) = \begin{cases} 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Hence, F is an antidirivative of f. However, f is not continuous at the point 0. Indeed, $\lim\limits_{x \longrightarrow 0} f(x) = \lim\limits_{x \longrightarrow 0} \left(2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x} \right)$ *does'not exist, since* $\lim\limits_{x \longrightarrow 0} \cos \dfrac{1}{x}$ *does not exist.*

1.2.2 Indefinite integral

Definition 30 *Let the function $f : [a,b] \longrightarrow \mathbb{R}$. The set of all antiderivatives of the function f is called the indefinite integral of f and is denoted by* $\int f(x)dx$.

Thus, if F is any antiderivative of f, we have

$$\int f(x)dx = \{F(x) + C,\ C \in \mathbb{R}\}.$$

We will write

$$\int f(x)dx = F(x) + C,\ C \in \mathbb{R}.$$

Theorem 31 ***(Properties of the indefinite integral)***
If f and g have antiderivatives, then $(f+g)$ and λf $(\lambda \in \mathbb{R})$ also have antiderivatives, and we have

1) $\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx.$

2) $\int \lambda f(x)dx = \lambda \int f(x)dx.$

Example 32 1) $\int \cos(x)dx = \sin(x) + C,\ C \in \mathbb{R}.$

2) $\int \frac{1}{x}dx = \ln|x| + C,\ C \in \mathbb{R}.$

3) $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ C \in \mathbb{R},\ \ n \neq -1.$

4) $\int e^{3x}dx = \frac{e^{3x}}{3} + C,\ C \in \mathbb{R}.$

5) $\int \sin^2(x)dx = \frac{1}{2}\int (1-\cos(2x))dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C,\ \ C \in \mathbb{R}.$

Remark 33 *Let f be a continuous function on $[a,b]$ and let two derivable functions $u, v : [\alpha, \beta] \longrightarrow [a,b]$. Then, the function $g(x) = \int\limits_{u(x)}^{v(x)} f(t)dt$ is derivable and we have $g'x) = f(v(x)).v'(x) - f(u(x)).u'(x)$.*

Indeed, let F be an antiderivative of f (i.e. $F' = f$).

$g(x) = F(v(x)) - F(u(x)) \Longrightarrow g'x) = F'(v(x)).v'(x) - F'(u(x)).u'(x)$,

then, $g'x) = f(v(x)).v'(x) - f(u(x)).u'(x)$.

1.3 General methods of integration

1.3.1 Integration by parts

Theorem 34 *Let u and v be two continuously derivable functions on $[a,b]$. Then, we have*

$$\int\limits_a^b u(x).v'(x)dx = [u(x).v(x)]_a^b - \int\limits_a^b u'(x).v(x)dx,$$

where $[u(x).v(x)]_a^b = u(b).v(b) - u(a).v(a)$.

Indeed, $(u.v)' = u.v' + u'.v$,

then, $\int_a^b (u(x).v(x))'dx = [u(x).v(x)]_a^b = \int_a^b u(x).v'(x)dx + \int_a^b u'(x).v(x)dx$,

hence, $\int_a^b u(x).v'(x)dx = [u(x).v(x)]_a^b - \int_a^b u'(x).v(x)dx$.

Example 35 *Calculate* $I = \int_0^1 \arctan(x)dx$.

We set

$$\begin{cases} u(x) = \arctan(x) \\ v'(x) = 1 \end{cases} \Longrightarrow \begin{cases} u'(x) = \dfrac{1}{1+x^2}, \\ v(x) = x, \end{cases}$$

then,

$$I = \int_0^1 \arctan(x)dx = [x.\arctan(x).]_0^1 - \int_0^1 \frac{x}{1+x^2}dx = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx,$$

$$I = \frac{\pi}{4} - \frac{1}{2}\left[\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln(2).$$

Example 36 *Calculate* $J = \int x^2 \ln(x)dx$.

We set

$$\begin{cases} u(x) = \ln(x) \\ v'(x) = x^2 \end{cases} \Longrightarrow \begin{cases} u'(x) = \dfrac{1}{x}, \\ v(x) = \dfrac{x^3}{3}, \end{cases}$$

$$J = \int x^2 \ln(x)dx = \frac{x^3}{3}\ln(x) - \int \frac{x^2}{3}dx = \frac{x^3}{3}\ln(x) - \frac{x^3}{9} + C, \quad C \in \mathbb{R}.$$

1.3.2 Change of variable

Theorem 37 *Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function and $\varphi : [\alpha,\beta] \longrightarrow [a,b]$ be a continuously derivable function such that $\varphi(\alpha) = a$ and $\varphi(\beta) = b$.*

Then, the function $g : [\alpha,\beta] \longrightarrow \mathbb{R}$ such that $g(t) = f(\varphi(t)).\varphi'(t)$ is integrable on $[\alpha,\beta]$ and we have

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t)).\varphi'(t)dt.$$

Remark 38 *We set* $x = \varphi(t) \Longrightarrow x' = \varphi'(t) \Longrightarrow \dfrac{dx}{dt} = \varphi'(t)$, *then* $dx = \varphi'(t)dt$.

Example 39 *Calculate* $I = \int\limits_0^1 \sqrt{1-x^2}dx$.

We set $x = \sin t \Longrightarrow dx = \cos t.dt$,

if $x = 0 \Longrightarrow t = 0$,
if $x = 1 \Longrightarrow t = \frac{\pi}{2}$,

$$\text{then, } I = \int\limits_0^{\frac{\pi}{2}} \sqrt{\cos^2(t)}\cos(t)dt = \int\limits_0^{\frac{\pi}{2}} |\cos(t)|\cos(t)dt = \int\limits_0^{\frac{\pi}{2}} \cos^2(t)dt.$$

We have $cos(2t) = 2\cos^2(t) - 1 \Longrightarrow \cos^2(t) = \dfrac{\cos(2t)+1}{2}$,

$$\text{hence } I = \int\limits_0^{\frac{\pi}{2}} \cos^2(t)dt = \int\limits_0^{\frac{\pi}{2}} \frac{\cos(2t)+1}{2}dt = \frac{1}{2}\left[\frac{\sin(2t)}{2} + t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Example 40 *Calculate* $J = \int \dfrac{x}{\sqrt{x+1}}dx$.

We set $t = \sqrt{x+1} \Longrightarrow x = t^2 - 1 \Longrightarrow dx = 2t.dt$,

$$\text{then } J = 2\int (t^2-1)dt = 2(\frac{t^3}{3} - t) + C, \quad C \in \mathbb{R},$$

$$\text{hence } J = 2\left(\frac{\sqrt{(x+1)^3}}{3} - \sqrt{x+1}\right) + C, \quad C \in \mathbb{R}.$$

Example 41 *We set* $t = \ln x \Longrightarrow dt = \dfrac{1}{x}dx$,

$$I_1 = \int \frac{\ln x}{x}dx = \int t.dt = \frac{t^2}{2} + C = \frac{\ln^2 x}{2} + C, \quad C \in \mathbb{R}.$$

$$I_2 = \int \frac{1}{x\ln x}dx = \int \frac{dt}{t} = \ln|t| + C = \ln|\ln x| + C, \quad C \in \mathbb{R}.$$

Example 42 $K = \int \sin^2 x \cos^3 x dx = \int \sin^2 x(1-\sin^2 x)\cos x dx$

$$= \int (\sin^2 x - \sin^4 x)\cos x.dx,$$

we set $t = \sin x \Longrightarrow dt = \cos x.dx$,

$$K = \int (t^2 - t^4)dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C, \quad C \in \mathbb{R}.$$

1.3.3 Partial fraction decomposition

Example 43 *Calculate* $I = \int \frac{1}{x(x+1)(x+2)}dx$.

We decompose the fraction into partial fractions :

$$\begin{aligned} \frac{1}{x(x+1)(x+2)} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \\ &= \frac{A(x+1)(x+2) + Bx(x+2) + Cx(x+1)}{x(x+1)(x+2)} \\ &= \frac{(A+B+C)x^2 + (3A+2B+C)x + 2A}{x(x+1)(x+2)}, \end{aligned}$$

by identification, we obtain

$$\begin{cases} A+B+C &= 0 \\ 3A+2B+C &= 0 \\ 2A &= 1 \end{cases} \Longrightarrow \begin{cases} A &= \frac{1}{2}, \\ B &= -1, \\ C &= \frac{1}{2}, \end{cases}$$

then $I = \int \frac{1}{x(x+1)(x+2)}dx = \int \left(\frac{1}{2}(\frac{1}{x}) - \frac{1}{x+1} + \frac{1}{2}(\frac{1}{x+2})\right) dx,$

hence $I = \frac{1}{2}\ln|x| - \ln|x+1| + \frac{1}{2}\ln|x+2| + C, \quad C \in \mathbb{R}.$

Example 44 *Calculate* $I = \int \frac{1}{x(1+x^2)}dx$.

We decompose the fraction into partial fractions :

$$\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+(Bx+C)x}{x(1+x^2)} = \frac{(A+B)x^2 + Cx + A}{x(1+x^2)},$$

by identification, we obtain

$$\begin{cases} A+B &= 0 \\ C &= 0 \\ A &= 1 \end{cases} \Longrightarrow \begin{cases} A &= 1, \\ B &= -1, \\ C &= 0, \end{cases}$$

then $I=\int \frac{1}{x(1+x^2)}dx=\int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx=\ln|x|-\frac{1}{2}\ln(1+x^2)+C,$ $C\in\mathbb{R}$.

Remark 45 1) *If the fraction* $\frac{P(x)}{Q(x)}$ *Is such that the degree of* P *is less than the degree of* Q *(i.e. a proper fraction), we perform the decomposition of the fraction into partial fractions (if possible).*

2) *If the fraction* $\frac{P(x)}{Q(x)}$ *Is such that the degree of* P *is greater than or equal to the degree of* Q*, we first perform the Euclidean division, we obtain* $\frac{P(x)}{Q(x)}=A(x)+\frac{R(x)}{Q(x)}$ *such that the deg* $R(x)$ *is less than the deg* $Q(x)$*.*

Then, we decompose the fraction $\frac{R(x)}{Q(x)}$ *into partial fractions (if possible).*

Example 46 *Calculate* $I=\int\frac{x^3}{x^2-1}dx$.

$I=\int\left(x+\frac{x}{x^2-1}\right)dx=\frac{x^2}{2}+\frac{1}{2}\ln\left|x^2-1\right|+C,\quad C\in\mathbb{R}.$

1.3.4 Antiderivatives of rational functions

1) $I=\int\frac{1}{x^2+a^2}dx,\qquad a\neq 0.$

$$I=\int\frac{1}{a^2(\frac{x^2}{a^2}+1)}dx=\frac{1}{a^2}\int\frac{1}{(\frac{x}{a})^2+1}dx=\frac{1}{a}\arctan(\frac{x}{a})+C,\qquad C\in\mathbb{R}.$$

(We can make the change of variable $t=\frac{x}{a}$.)

2) $J=\int\frac{1}{x^2-a^2}dx,\qquad a\neq 0.$

$$J=\int\frac{1}{(x+a)(x-a)}dx.$$

We decompose the fraction into partial fractions :

$$\frac{1}{(x+a)(x-a)}=\frac{A}{x+a}+\frac{B}{x-a}=\frac{(A+B)x-Aa+Ba}{(x-a)(x+a)},$$

by identification, we obtain

$$\begin{cases} A+B &= 0\\ -Aa+Ba &= 1\end{cases}\Longrightarrow\begin{cases} A &= -\frac{1}{2a},\\ B &= \frac{1}{2a},\end{cases}$$

then

$$\begin{aligned} J &= \int \frac{1}{(x+a)(x-a)}dx = -\frac{1}{2a}\int \frac{1}{x+a}dx + \frac{1}{2a}\int \frac{1}{x-a}dx, \\ &= -\frac{1}{2a}\ln|x+a| + \frac{1}{2a}\ln|x-a| + C, \quad C \in \mathbb{R} \\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C, \quad C \in \mathbb{R}. \end{aligned}$$

Calculation of integrals of the type : $I = \int \frac{mx+n}{ax^2+bx+c}dx$, $a \neq 0$.

1) If $m = 0$, then $I = \int \frac{n}{ax^2+bx+c}dx$

- if $ax^2+bx+c=0$ has two real roots, we factor, then we decompose the fraction into partial fractions.

- If $ax^2+bx+c=0$ has a double root, we factor, then we integrate.

- If $ax^2+bx+c=0$ has no real roots, we rewrite this polynomial in the form X^2+A^2 ou X^2-A^2.

Example 47 *Calculate* $I = \int \frac{1}{x^2+2x+5}dx.$

$I = \int \frac{1}{(x+1)^2+4}dx = \frac{1}{4}\int \frac{1}{(\frac{x+1}{2})^2+1}dx,$

we set $t = \frac{x+1}{2} \Longrightarrow dt = \frac{1}{2}dx,$

then

$$\begin{aligned} I &= \frac{1}{2}\int \frac{1}{t^2+1}dt = \frac{1}{2}\arctan(t) + C, \quad C \in \mathbb{R}, \\ &= \frac{1}{2}\arctan(\frac{x+1}{2}) + C, \quad C \in \mathbb{R}. \end{aligned}$$

2) *If* $m \neq 0$, *we write the integral in the following form*

$$\begin{aligned} \int \frac{mx+n}{ax^2+bx+c}dx &= \int \frac{\frac{m}{2a}(2ax+b) + (n - \frac{mb}{2a})}{ax^2+bx+c}dx \\ &= \frac{m}{2a}\int \frac{(2ax+b)}{ax^2+bx+c}dx + (n-\frac{mb}{2a})\int \frac{1}{ax^2+bx+c}dx \\ &= \frac{m}{2a}\ln\left|ax^2+bx+c\right| + (n-\frac{mb}{2a})\int \frac{1}{ax^2+bx+c}dx. \end{aligned}$$

Example 48 *Calculate* $I = \int \frac{x+1}{x^2+x+1}dx.$

$$\begin{aligned} I &= \frac{1}{2}\int \frac{2(x+1)}{x^2+x+1}dx = \frac{1}{2}\int \left(\frac{2x+1}{x^2+x+1} + \frac{1}{x^2+x+1}\right) dx \\ &= \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{2}\int \frac{1}{x^2+x+1}dx. \end{aligned}$$

$$\begin{aligned} \int \frac{1}{x^2+x+1}dx &= \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}dx = \frac{4}{3}\int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}dx \\ &= \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C, \quad C \in \mathbb{R}. \end{aligned}$$

(*we can make the change of variable* $t = \frac{2x+1}{\sqrt{3}}$).

Therefore, $I = \frac{1}{2}\ln\left|x^2+x+1\right| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C, \quad C \in \mathbb{R}.$

Calculation of integrals of the type : $\int R(\sin x, \cos x)dx,$

Such that R is a rational function.

On pose $t = \tan\frac{x}{2}$,

$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right) \Longrightarrow dx = \frac{2dt}{1+t^2},$

$\sin x = \sin 2(\frac{x}{2}) = 2\sin\frac{x}{2}.\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}.\cos\frac{x}{2}}{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2},$

thus, $\sin x = \frac{2t}{1+t^2}.$

$\cos x = \cos 2(\frac{x}{2}) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right). = \frac{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2},$

therefore, $\cos x = \frac{1-t^2}{1+t^2}.$

Example 49 *Calculate* $I = \int \frac{1}{\sin x} dx.$

We set $t = \tan \frac{x}{2}$,

then, $dx = \frac{2dt}{1+t^2}$ *and* $\sin x = \frac{2t}{1+t^2}$.

$$I = \int \frac{1}{\frac{2t}{1+t^2}} \left(\frac{2dt}{1+t^2} \right) = \int \frac{dt}{t} = \ln |t| + C, \quad C \in \mathbb{R},$$

hence, $I = \ln \left| \tan \frac{x}{2} \right| + C, \quad C \in \mathbb{R}.$

Example 50 *Calculate* $I = \int \frac{1}{1 + \sin x + \cos x} dx.$

We set $t = \tan \frac{x}{2}$,

then, $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ *et* $\cos x = \frac{1-t^2}{1+t^2}$.

$$I = \int \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \left(\frac{2dt}{1+t^2} \right) = \int \frac{dt}{1+t} = \ln |1+t| + C, \quad C \in \mathbb{R},$$

hence, $I = \ln \left| 1 + \tan \frac{x}{2} \right| + C, \quad C \in \mathbb{R}.$

Calculation of integrals of the type : $\int R(e^x) dx.$

such that R is a rational function.

We set $t = e^x$,

$$\frac{dt}{dx} = e^x = t \Longrightarrow dx = \frac{dt}{t}.$$

Example 51 *Calculate* $I = \int \frac{dx}{1 + e^x}.$

We set $t = e^x \Longrightarrow dx = \frac{dt}{t}$,

$$I = \int \frac{dt}{t(1+t)} = \int \left(\frac{1}{t} - \frac{1}{1+t} \right) dt,$$

$I = \ln |t| - \ln |1+t| + C, \quad C \in \mathbb{R},$

then, $I = \ln |e^x| - \ln |1 + e^x| + C, \quad C \in \mathbb{R},$

hence, $I = x - \ln(1 + e^x) + C, \quad C \in \mathbb{R}.$

1.4 Exercises

Exercise 52 1) *Calculate* $I = \displaystyle\int \frac{t}{t^2+2t-3}dt.$

2) *Deduce* $J = \displaystyle\int \frac{e^x}{e^x - 3e^{-x}+2}dx.$

Solution :
1) $I = \displaystyle\int \frac{t}{t^2+2t-3}dt = \int \frac{t}{(t+3)(t-1)}dt.$

We decompose the fraction into partial fractions :

$\dfrac{t}{(t+3)(t-1)} = \dfrac{A}{t+3} + \dfrac{B}{t-1},$

we obtain

$I = \displaystyle\int \left(\frac{3}{4(t+3)} + \frac{1}{4(t-1)}\right) dt,$

then, $I = \dfrac{3}{4}\ln|t+3| + \dfrac{1}{4}\ln|t-1| + C, \quad C \in \mathbb{R}.$

2) $J = \displaystyle\int \frac{e^x}{e^x - 3e^{-x}+2}dx.$

We set $t = e^x$,

$\dfrac{dt}{dx} = e^x = t \Longrightarrow dx = \dfrac{dt}{t},$

$J = \displaystyle\int \frac{t}{t-3t^{-1}+2}\frac{dt}{t} = \int \frac{t}{t^2+2t-3}dt = I,$

then, $J = \dfrac{3}{4}\ln|t+3| + \dfrac{1}{4}\ln|t-1| + C, \quad C \in \mathbb{R},$

hence, $J = \dfrac{3}{4}\ln(e^x+3) + \dfrac{1}{4}\ln|e^x-1| + C, \quad C \in \mathbb{R}.$

Exercise 53 1) *Calculate the integral* $I = \displaystyle\int \frac{dx}{x(x^3+1)}.$

2) *Deduce the integral* $J = \displaystyle\int \frac{x^2\ln x}{(x^3+1)^2}dx.$

Solution :
1) $I = \displaystyle\int \frac{dx}{x(x^3+1)}.$

We decompose the fraction into partial fractions :

$$\begin{aligned}\frac{1}{x(x^3+1)} &= \frac{1}{x(x+1)(x^2-x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} \\ &= \frac{A(x^3+1) + Bx(x^2-x+1) + (Cx+D)x(x+1)}{x(x+1)(x^2-x+1)} \\ &= \frac{(A+B+C)x^3 + (-B+C+D)x^2 + (B+D)x + A}{x(x^3+1)},\end{aligned}$$

by identification, we obtain

$$\begin{cases} A+B+C &= 0 \\ -B+C+D &= 0 \\ B+D &= 0 \\ A &= 1 \end{cases} \Longleftrightarrow \begin{cases} A &= 1, \\ B &= -\dfrac{1}{3}, \\ C &= -\dfrac{2}{3}, \\ D &= \dfrac{1}{3}, \end{cases}$$

then

$$I = \int \frac{dx}{x(x^3+1)} = \int \left(\frac{1}{x} - \frac{1}{3}\frac{1}{x+1} - \frac{1}{3}\frac{2x-1}{x^2-x+1} \right) dx,$$

$$I = \int \frac{1}{x}dx - \frac{1}{3}\int \frac{1}{x+1}dx - \frac{1}{3}\int \frac{2x-1}{x^2-x+1}dx,$$

$$I = \ln|x| - \frac{1}{3}\ln|x+1| - \frac{1}{3}\ln\left|x^2-x+1\right| + C, \ \ C \in \mathbb{R},$$

hence, $I = \ln|x| - \dfrac{1}{3}\ln\left|x^3+1\right| + C, \ \ C \in \mathbb{R}$.

2) We deduce the integral $J = \displaystyle\int \frac{x^2 \ln x}{(x^3+1)^2}dx$.

We perform integration by parts, we set

$$\begin{cases} U &= \ln x \\ V' &= \dfrac{x^2}{(x^3+1)^2} \end{cases} \Longrightarrow \begin{cases} U' &= \dfrac{1}{x}, \\ V &= -\dfrac{1}{3}\dfrac{1}{x^3+1}, \end{cases}$$

then

$$J = \int \frac{x^2 \ln x}{(x^3+1)^2}dx = -\frac{\ln x}{3(x^3+1)} + \frac{1}{3}\int \frac{1}{x(x^3+1)}dx,$$

$$J = -\frac{\ln x}{3(x^3+1)} + \frac{1}{3}I,$$

hence $J = -\dfrac{\ln x}{3(x^3+1)} + \dfrac{1}{3}\ln|x| - \dfrac{1}{9}\ln\left|x^3+1\right| + C, \ \ C \in \mathbb{R}$.

Exercise 54 1) *Calculate the integral* $I = \displaystyle\int \frac{2}{(1+t)(1+t^2)}dt$.

2) *Deduce the integral* $J = \displaystyle\int \frac{\sin x}{1+\sin x - \cos x}dx$.

Solution :
1) $I = \displaystyle\int \frac{2}{(1+t)(1+t^2)}dt$.
We decompose the fraction into partial fractions :

$$\begin{aligned}\frac{2}{(1+t)(1+t^2)} &= \frac{A}{1+t} + \frac{Bt+C}{1+t^2} = \frac{A(1+t^2)+(Bt+C)(1+t)}{(1+t)(1+t^2)} \\ &= \frac{(A+B)t^2+(B+C)t+A+C}{(1+t)(1+t^2)},\end{aligned}$$

by identification, we obtain

$$\begin{cases} A+B &= 0 \\ B+C &= 0 \\ A+C &= 2 \end{cases} \Longleftrightarrow \begin{cases} A &= 1, \\ B &= -1, \\ C &= 1, \end{cases}$$

then

$$I = \int \frac{2}{(1+t)(1+t^2)}dt = \int \left(\frac{1}{1+t} + \frac{-t+1}{1+t^2}\right)dt,$$

$$I = \int \left(\frac{1}{1+t} - \frac{t}{1+t^2} + \frac{1}{1+t^2}\right)dt = \int \frac{1}{1+t}dt - \int \frac{t}{1+t^2}dt + \int \frac{1}{1+t^2}dt,$$

hence, $I = \ln|1+t| - \frac{1}{2}\ln\left(1+t^2\right) + \arctan t + C, \ \ C \in \mathbb{R}$.

2) We deduce the integral $J = \displaystyle\int \frac{\sin x}{1+\sin x - \cos x}dx$.

We make a change of variable, we set $t = \tan\frac{x}{2}$,

with this change of variable, we obtain

$\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$,

then

$$J = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2} = \int \frac{2}{(1+t)(1+t^2)}dt = I,$$

hence $J = \ln|1+t| - \frac{1}{2}\ln\left(1+t^2\right) + \arctan t + C, \ C \in \mathbb{R}$, thus $J = \ln\left|1+\tan\frac{x}{2}\right| - \frac{1}{2}\ln\left(1+\tan^2\left(\frac{x}{2}\right)\right) + \frac{x}{2} + C,\ C \in \mathbb{R}.$

Exercise 55 1) *Calculate* $I = \int_0^1 \ln(x+t)dx \quad , \quad t \in]0,+\infty[.$

2) *Deduce* $J_n = \int_0^1 \ln[(x+1)(x+2).....(x+n)]dx \quad , \quad n \in \mathbb{N}^*.$

3) *Calculate* $\lim\limits_{n\to+\infty} \frac{J_n}{(n+1)^2}.$

Solution :

1) $I = \int_0^1 \ln(x+t)dx, \quad t \in]0,+\infty[.$

We perform integration by parts, we set

$$\begin{cases} U = \ln(x+t) \\ V' = 1 \end{cases} \Longrightarrow \begin{cases} U' = \frac{1}{x+t}, \\ V = x, \end{cases}$$

then

$$J = [x\ln(x+t)]_{x=0}^{x=1} - \int_0^1 \frac{x}{x+t}dx = \ln(1+t) - \int_0^1 \frac{x+t-t}{x+t}dx$$

$$= \ln(1+t) - \int_0^1 \left(1 - \frac{t}{x+t}\right)dx = \ln(1+t) - [x - t\ln(x+t)]_{x=0}^{x=1}$$

$$= \ln(1+t) - 1 + t\ln(1+t) - t\ln t,$$

hence, $J = (1+t)\ln(1+t) - t\ln t - 1.$

2) We deduce $J_n = \int_0^1 \ln[(x+1)(x+2).....(x+n)]dx, \quad n \in \mathbb{N}^*.$

$$J_n = \int_0^1 \ln[(x+1)(x+2).....(x+n)]dx$$

$$= \int_0^1 \ln(x+1)dx + \int_0^1 \ln(x+2)dx + ... + \int_0^1 \ln(x+n)dx.$$

According to the first question, we obtain

$J_n = (2\ln 2 - 1) + (3\ln 3 - 2\ln 2 - 1) + ... + ((n+1)\ln(n+1) - n\ln n - 1),$

then $J_n = (n+1)\ln(n+1) - n.$

3) we calculate $\lim\limits_{n\to+\infty} \frac{J_n}{(n+1)^2}.$

$$\lim_{n\to+\infty}\frac{J_n}{(n+1)^2} = \lim_{n\to+\infty}\frac{(n+1)\ln(n+1)-n}{(n+1)^2},$$

$$\lim_{n\to+\infty}\frac{J_n}{(n+1)^2} = \lim_{n\to+\infty}\left(\frac{\ln(n+1)}{(n+1)} - \frac{n}{(n+1)^2}\right) = 0.$$

Chapter 2

Differential equations of the first and second order

2.1 Differential equations of the first order

Definition 56 *A differential equation of the first order is defined as any equation of the form*

$$y' = f(x, y) \qquad (I)$$

where $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function and y is a function of the variable x.

The solution of the differential equation :

Let I be an interval of $\mathbb{R}$. The function $y : I \longrightarrow \mathbb{R}$ is a solution of the differential equation (I) if it satisfies the following conditions :

1) The graph of y, $G_y \subset D$, i.e. $\forall x \in I, (x, y(x)) \in D$.

2) y is a derivable function and we have $\forall x \in I, y'(x) = f(x, y(x))$.

In this chapter, we will study five types of first-order differential equations.

2.1.1 Differential equations with separable variables

Definition 57 *Differential equations with separable variables are written in the following form :*

$$y' = f(x).g(y) \qquad (1)$$

where $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ are two continuous functions with I and J two intervals of $\mathbb{R}$.

Solving method :

We have $y' = \dfrac{dy}{dx} = f(x)g(y)$,

then, $\dfrac{dy}{g(y)} = f(x)dx$,

hence, $\displaystyle\int \frac{dy}{g(y)} = \int f(x)dx$, with $g(y) \neq 0$.

We integrate to find $y = \varphi(x)$ which is the solution of the differential equation (1).

Exercise 58 *Solve the following differential equations :*

1) $(x+1)y' + y = 0$.

2) $y' \sin x - y \cos x = 0$.

3) $y' + \dfrac{xy}{1-x^2} = 0$, *satisfying* $y\ (0) = 1$.

Solution :
1) $(x+1)y' + y = 0$.....(E).

Remark : $y = 0$ is a solution of (E).

If $y \neq 0$, $(x+1)y' + y = 0 \Longleftrightarrow y' = -\dfrac{1}{x+1}y$

$\Longleftrightarrow \dfrac{dy}{dx} = -\dfrac{1}{x+1}y \Longleftrightarrow \dfrac{dy}{y} = -\dfrac{1}{x+1}dx$,

then

$$\int \frac{dy}{y} = -\int \frac{1}{x+1}dx \Longrightarrow \ln|y| = -\ln|x+1| + C = \ln\frac{1}{|x+1|} + C, \quad C \in \mathbb{R},$$

hence, $|y| = \dfrac{1}{|x+1|}e^C \Longrightarrow y = \pm e^C \dfrac{1}{x+1} \Longrightarrow y = \dfrac{K}{x+1}$, $\quad K \in \mathbb{R}^*$.

Since $y = 0$ is a solution of (E), then the general solution of (E) is

$y = \dfrac{K}{x+1}$, $\quad K \in \mathbb{R}$.

*****

2) $y' \sin x - y \cos x = 0$.....(F).

Remark : $y = 0$ is a solution of (F).

If $y \neq 0$, $y' \sin x - y \cos x = 0 \Longleftrightarrow y' = \dfrac{\cos x}{\sin x}y$

$\iff \dfrac{dy}{dx} = \dfrac{\cos x}{\sin x} y \iff \dfrac{dy}{y} = \dfrac{\cos x}{\sin x} dx,$

then

$\displaystyle\int \frac{dy}{y} = \int \frac{\cos x}{\sin x} dx \Longrightarrow \ln|y| = \ln|\sin x| + C, \quad C \in \mathbb{R},$

hence, $|y| = |\sin x| \, e^C \Longrightarrow y = \pm e^C \sin x \Longrightarrow y = K \sin x, \;\; K \in \mathbb{R}^*.$

Since $y = 0$ is a solution of (F), then the general solution of (F) is

$y = K \sin x, \;\; K \in \mathbb{R}.$

*****

3) $y' + \dfrac{xy}{1-x^2} = 0.....(G).$

Remark : $y = 0$ is a solution of (G).

If $y \neq 0$, $y' + \dfrac{xy}{1-x^2} = 0 \iff y' = -\dfrac{x}{1-x^2} y$

$\iff \dfrac{dy}{dx} = -\dfrac{x}{1-x^2} y \iff \dfrac{dy}{y} = \dfrac{x}{x^2-1} dx,$

then

$\displaystyle\int \frac{dy}{y} = \int \frac{x}{x^2-1} dx \Longrightarrow \ln|y| = \frac{1}{2} \ln\left|x^2-1\right| + C, \quad C \in \mathbb{R},$

hence, $|y| = \sqrt{|x^2-1|} e^C \Longrightarrow y = \pm e^C \sqrt{|x^2-1|} \Longrightarrow y = K\sqrt{|x^2-1|}, \;\; K \in \mathbb{R}^*.$

Since $y = 0$ is a solution of (G), then the general solution of (G) is

$y = K\sqrt{|x^2-1|}, \;\; K \in \mathbb{R}.$

We look for the solution that satisfies the condition $y(0) = 1$.

$y(0) = 1 \iff K = 1,$

thus, $y = \sqrt{|x^2-1|}.$

2.1.2 Homogeneous differential equations

Definition 59 *The homogeneous differential equations are written in the following form :*

$$y' = f\left(\frac{y}{x}\right) \qquad (2)$$

where $f : I \longrightarrow \mathbb{R}$ *is a continuous function.*

Solving method :

$y' = f\left(\frac{y}{x}\right)$,

we set $t = \frac{y}{x} \Longleftrightarrow y = tx$, then $y' = t'x + t$.

We substitute into the equation (2) :

$t'x + t = f(t) \Longleftrightarrow t'x = f(t) - t \Longleftrightarrow t' = (f(t) - t)\frac{1}{x}$,

thus, we obtain a differential equation with separable variables,

$\frac{dt}{dx} = (f(t) - t)\frac{1}{x}$.

If $f(t) - t \neq 0$, then we have $\int \frac{dt}{f(t) - t} = \int \frac{1}{x}dx = \ln|x| + C$,

by integrating, we obtain $t = \varphi(x)$ and then, $y = x\varphi(x)$ is the solution of the equation (2).

Singular solutions of the homogeneous equation :
If $f(t) - t = 0$, then we have:

let t_0 be a root of this equation, then $t = t_0$ is a solution to the differential equation
$t'x = f(t) - t$.

Indeed, $f(t_0) - t_0 = 0$ and since t_0 is a constant, then $t' = (t_0)' = 0$,

hence, $y = xt_0$ is a solution of the equation (2).

These solutions are called **the singular solutions of the homogeneous equation.**

Exercise 60 *Solve the following differential equations :*
1) $x\ (y' - \frac{y}{x}) - y + x = 0$.

2) $y'(2\sqrt{xy} - x) + y = 0,$ *on* $]0, +\infty[$ *satisfying* $y(1) = 1$.

Solution :
1) $x\ (y' - \frac{y}{x}) - y + x = 0....(E)$.

$x\ (y' - \frac{y}{x}) - y + x = 0 \Longleftrightarrow y' = 2\frac{y}{x} - 1$.

We set $t = \frac{y}{x} \Longleftrightarrow y = tx$, then $y' = t'x + t$,

$t'x + t = 2\frac{y}{x} - 1 = 2t - 1 \Longrightarrow t'x = t - 1$, thus $t' = \frac{1}{x}(t-1)$: It is a differential equation with separable variables,

so, $\frac{dt}{dx} = \frac{1}{x}(t-1) \Longleftrightarrow \frac{dt}{t-1} = \frac{dx}{x}$ if $t-1 \neq 0$,

then, $\int \frac{dt}{t-1} = \int \frac{dx}{x} \Longrightarrow \ln|t-1| = \ln|x| + C, \quad C \in \mathbb{R}$,

thus, $|t-1| = |x|\, e^C \Longrightarrow t-1 = \pm e^C x \Longrightarrow t = Kx+1, \;\; K \in \mathbb{R}^*$.

Therefore, $y = tx = Kx^2 + x, \;\; K \in \mathbb{R}^*$.

The singular solutions of (E) :

If $t-1=0 \Longrightarrow t=1 \Longrightarrow \frac{y}{x} = 1$,

then $y = x$: It is the singular solution of the equation (E).

*****

2) $y'(2\sqrt{xy} - x) + y = 0$, on $]0, +\infty[$ satisfying $y(1) = 1....(F)$.

$$y'(2\sqrt{xy} - x) + y = 0 \Longleftrightarrow y'\left(2\sqrt{\frac{y}{x}} - 1\right) + \frac{y}{x} = 0$$

$$\Longleftrightarrow y' = -\frac{\frac{y}{x}}{2\sqrt{\frac{y}{x}} - 1} \quad \text{with} \quad 2\sqrt{\frac{y}{x}} - 1 \neq 0,$$

we set $t = \frac{y}{x} \Longleftrightarrow y = tx$, then $y' = t'x + t$,

$$t'x + t = -\frac{\frac{y}{x}}{2\sqrt{\frac{y}{x}} - 1} = \frac{-t}{2\sqrt{t} - 1} \Longrightarrow t'x = \frac{-2t\sqrt{t}}{2\sqrt{t} - 1},$$

thus, $t' = \frac{1}{x}\left(\frac{-2t\sqrt{t}}{2\sqrt{t}-1}\right)$: t is a differential equation with separable variables,

then, $\frac{dt}{dx} = \frac{1}{x}\left(\frac{-2t\sqrt{t}}{2\sqrt{t}-1}\right) \Longleftrightarrow \left(\frac{2\sqrt{t}-1}{-2t\sqrt{t}}\right) dt = \frac{dx}{x}$ if $t \neq 0$,

hence

$$\int \left(\frac{2\sqrt{t}-1}{-2t\sqrt{t}}\right) dt = \int \frac{dx}{x} \Longrightarrow \int \left(\frac{-1}{t} + \frac{1}{2t\sqrt{t}}\right) dt = \int \frac{dx}{x}$$

$$\Longrightarrow -\ln t - \frac{1}{\sqrt{t}} = \ln x + C, \quad C \in \mathbb{R}$$

$$\Longrightarrow \ln(xt) = -\frac{1}{\sqrt{t}} - C \Longrightarrow xt = e^{-\frac{1}{\sqrt{t}} - C},$$

therefore, $y = e^{-\sqrt{\frac{x}{y}} - C}$.

The singular solution of the equation (F) :

if $t = 0 \Longrightarrow y = 0$: It is the singular solution of the equation (F).

We seek the solution y that satisfies the condition $y(1) = 1$,

$y(1) = 1 \Longleftrightarrow e^{-1-C} = 1 \Longleftrightarrow C = -1$,

then, $y = e^{-\sqrt{\frac{x}{y}}+1}$.

2.1.3 Linear differential equations of the first order

Definition 61 *The linear differential equations of the first order are written in the following form :*

$$y' + a(x)y = b(x) \qquad (E)$$

where $a : I \longrightarrow \mathbb{R}$ *and* $b : I \longrightarrow \mathbb{R}$ *are two continuous functions.*

Solving method :

The general solution of (E) : $y_G = y_p + y$,

where y_p is a particular solution of (E),

and y is the general solution of the equation without the right-hand side (E_0).

(E_0) : $y' + a(x)y = 0$: it is a differential equation with separable variables.

If the particular solution y_p is not obvious, we apply the method of variation of the constant, which involves replacing the constant K in the solution y of the equation without the right-hand side, by the function $K(x)$, then, we search for $K(x)$ by substituting into the equation (E).

Exercise 62 *Solve the following differential equations :*

1) $y' + xy = x$.

2) $y' - \dfrac{y}{x} = \ln x$.

3) $x(x^2 + 1)y' - 2y = x^3(x - 1)e^{-x}$.

Solution :

1) $y' + xy = x$......(E).

The general solution of (E) : $y_G = y_p + y$,

where y_p is a particular solution of (E),

and y is the general solution of the equation without the right-hand side :

$(E_0) : y' + xy = 0.$

We notice that $y_p = 1$ is a particular solution of (E).

We now seek the general solution of the equation without the right-hand side :

$(E_0) : y' + xy = 0$: it is a differential equation with separable variables.

We notice that $y = 0$ is a solution of (E_0).

If $y \neq 0$, $y' + xy = 0 \iff y' = -xy$

$$\iff \frac{dy}{dx} = -xy \iff \frac{dy}{y} = -xdx,$$

then, $\displaystyle\int \frac{dy}{y} = -\int xdx \Longrightarrow \ln|y| = -\frac{x^2}{2} + C, \quad C \in \mathbb{R}$

$$\Longrightarrow |y| = e^{-\frac{x^2}{2}} e^C \Longrightarrow y = \pm e^C e^{-\frac{x^2}{2}}$$

hence, $y = Ke^{-\frac{x^2}{2}}, \; K \in \mathbb{R}^*$.

Since $y = 0$ is a solution de (E_0), then the solution of (E_0) is

$y = Ke^{-\frac{x^2}{2}}, \; K \in \mathbb{R}$.

Therefore the general solution of (E) is given by

$y_G = y_p + y = 1 + Ke^{-\frac{x^2}{2}}, \quad K \in \mathbb{R}$.

*****

2) $y' - \dfrac{y}{x} = \ln x ...(F)$.

The general solution of (F) is : $y_G = y_p + y$,

where y_p is a particular solution of (F),

and y is the general solution of the equation without the right-hand side, $(F_0) : y' - \dfrac{y}{x} = 0$.

Since the particular solution y_p of (F) is not obvious, we first look for the general solution of the equation without the right-hand side :

$(F_0) : y' - \dfrac{y}{x} = 0$: it is a differential equation with separable variables.

We notice that $y = 0$ is a solution of (F_0).

If $y \neq 0$, $y' - \dfrac{y}{x} = 0 \iff y' = \dfrac{y}{x}$

$$\iff \frac{dy}{dx} = \frac{y}{x} \iff \frac{dy}{y} = \frac{dx}{x},$$

then, $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} \Longrightarrow \ln|y| = \ln|x| + C, \quad C \in \mathbb{R},$

hence, $|y| = |x| e^C \Longrightarrow y = \pm e^C x \Longrightarrow y = Kx, \quad K \in \mathbb{R}^*$.

Since $y = 0$ is a solution of (F_0), then the general solution of (F_0) is $y = Kx, \quad K \in \mathbb{R}$.

To find the particular solution, we apply **the method of variation of the constant (MVC)**. This method involves replacing the constant K with a function $K(x)$.

We set $y_G = K(x)x$ which is the general solution of (F),

then, $y'_G = K'(x)x + K(x)$.

We replace y_G and y'_G in the equation (F) to obtain $K'(x)$:

$$K'(x)x + K(x) - \frac{K(x)x}{x} = \ln x \Longrightarrow K'(x) = \frac{\ln x}{x},$$

then, $K(x) = \int \frac{\ln x}{x} dx$.

We set $U = \ln x \Longrightarrow dU = \frac{1}{x} dx$,

hence, $K(x) = \int \frac{\ln x}{x} dx = \int U dU = \frac{U^2}{2} + C = \frac{(\ln x)^2}{2} + C, \quad C \in \mathbb{R}$,

thus, $y_G = K(x)x = \frac{(\ln x)^2}{2} x + Cx$.

*****

3) $x(x^2 + 1)y' - 2y = x^3(x - 1)e^{-x}$......$(G)$.

Since the particular solution y_p of (G) is not obvious, we first look for the general solution of the equation without the right-hand side :

(G_0) : $x(x^2 + 1)y' - 2y = 0$: it is a differential equation with separable variables.

We notice that $y = 0$ is a solution of (G_0).

If $y \neq 0$, $x(x^2 + 1)y' - 2y = 0 \Longleftrightarrow y' = \frac{2}{x(x^2 + 1)} y$

$$\Longleftrightarrow \frac{dy}{dx} = \frac{2}{x(x^2 + 1)} y \Longleftrightarrow \frac{dy}{y} = \frac{2}{x(x^2 + 1)} dx.$$

By decomposing the fraction into partial fractions, we obtain

$$\int \frac{dy}{y} = \int \frac{2}{x(x^2 + 1)} dx = \int \left(\frac{2}{x} - \frac{2x}{x^2 + 1} \right) dx$$

$$\Longrightarrow \ln |y| = 2 \ln |x| - \ln(x^2 + 1) + C = \ln \frac{x^2}{x^2 + 1} + C, \quad C \in \mathbb{R},$$

hence, $y = \dfrac{Kx^2}{x^2+1}, \quad K \in \mathbb{R}.$

To find the particular solution, we apply the method of variation of the constant (MVC). This method involves replacing the constant K with a function $K(x)$.

We set $y_G = \dfrac{K(x)x^2}{x^2+1}$,

then, $y'_G = \dfrac{K'(x)x^2(x^2+1)+2xK(x)}{(x^2+1)^2}$.

We replace y_G and y'_G in the equation (G) to obtain $K'(x)$:

$x(x^2+1)\dfrac{K'(x)x^2(x^2+1)+2xK(x)}{(x^2+1)^2} - 2\dfrac{K(x)x^2}{x^2+1} = x^3(x-1)e^{-x}$,

then, $K'(x) = (x-1)e^{-x} \Longrightarrow K(x) = \displaystyle\int (x-1)e^{-x}dx.$

By performing integration by parts, we obtain

$K(x) = -xe^{-x} + C, \quad C \in \mathbb{R}$,

therefore, $y_G = \dfrac{K(x)x^2}{x^2+1} = \dfrac{(-xe^{-x}+C)x^2}{x^2+1} = \dfrac{-x^3e^{-x}}{x^2+1} + \dfrac{Cx^2}{x^2+1}$.

Exercise 63 1) *Calculate the following integral :*

$$I = \int \frac{dx}{(1+x^2)(1+x)}.$$

2) *Deduce the following 'integral :*

$$J = \int \frac{\arctan x}{(1+x)^2}dx.$$

3) *Solve the following differential equation, specifying its type :*

$$(E) : (x+1)y' + y = \frac{\arctan x}{(1+x)^2}.$$

Solution :

1) Let the integral

$$I = \int \frac{dx}{(1+x^2)(1+x)},$$

decomposing the fraction into partial fractions, we obtain :

$$\begin{aligned} I &= \int \frac{dx}{(1+x^2)(1+x)} = \int \left(\frac{\frac{1}{2}}{1+x} + \frac{-\frac{1}{2}x+\frac{1}{2}}{1+x^2} \right) dx \\ &= \frac{1}{2}\ln|1+x| - \frac{1}{4}\ln(1+x^2) + \frac{1}{2}\arctan x + C, \quad C \in \mathbb{R}. \end{aligned}$$

2) We deduce the integral

$$J = \int \frac{\arctan x}{(1+x)^2} dx.$$

By performing integration by parts, we obtain

$$J = \int \frac{\arctan x}{(1+x)^2} dx = -\frac{\arctan x}{1+x} + \int \frac{dx}{(1+x^2)(1+x)} = -\frac{\arctan x}{1+x} + I,$$

then

$$J = -\frac{\arctan x}{1+x} + \frac{1}{2}\ln|1+x| - \frac{1}{4}\ln(1+x^2) + \frac{1}{2}\arctan x + C, \quad C \in \mathbb{R}.$$

3) Let the following differential equation :

$$(E) : (x+1)y' + y = \frac{\arctan x}{(1+x)^2},$$

it is a linear differential equation of the first order, which we solve using the method of variation of the constant. We find

$y = \dfrac{K(x)}{1+x}$ with $K(x) = J$,

hence

$$y = -\frac{\arctan x}{(1+x)^2} + \frac{1}{2}\frac{\ln|1+x|}{1+x} - \frac{1}{4}\frac{\ln(1+x^2)}{1+x} + \frac{1}{2}\frac{\arctan x}{1+x} + \frac{C}{1+x}, \quad C \in \mathbb{R}.$$

2.1.4 Bernoulli differential equations

Definition 64 *Bernoulli differential equations are written in the following form :*

$$y' + a(x)y = b(x)y^k, \quad (E) \qquad k \in \mathbb{R} \setminus \{0,1\},$$

where $a : I \longrightarrow \mathbb{R}$ and $b : I \longrightarrow \mathbb{R}$ are two continuous functions.

Solving method :

We check if $y = 0$ is a solution of the equation (E).

If $y \neq 0$, we divide the equation (E) by y^k, we obtain

$y'y^{-k} + a(x)y^{1-k} = b(x) \quad (E').$

We make the following variable change :

$Z = y^{1-k} \Longrightarrow Z' = (1-k)y'y^{-k}$,

then, we substitute into (E') to obtain

$\frac{Z'}{1-k} + a(x)Z = b(x)$: it is a linear differential equation.

If Z is a solution of this equation, then $y = Z^{\frac{1}{1-k}}$ is the solution of the equation (E).

Exercise 65 *Solve the following differential equations :*

1) $xy' + y = y^2 \ln x$.

2) $(1-x^3)y' + 3x^2y = -y^2$

Solution :

1) $xy' + y = y^2 \ln x$........(E).

Remark : $y = 0$ is a solution of (E).

If $y \neq 0$, we divide the equation (E) by y^2, we obtain

$xy'y^{-2} + y^{-1} = \ln x$........$(E')$.

We set $Z = y^{-1}$, then $Z' = -y'y^{-2}$.

We substitute into the equation (E'), we find

$-xZ' + Z = \ln x$...(E_ℓ) : it is a linear différential equation of order 1.

We solve this equation using the method of variation of the constant (see exercise 3), and we obtain

$Z = K(x)x = (\frac{1}{x}\ln x + \frac{1}{x} + C)x = \ln x + 1 + Cx, \qquad C \in \mathbb{R}$,

then, $y = \frac{1}{Z} = \frac{1}{\ln x + 1 + Cx}$.

*****

2) $(1-x^3)y' + 3x^2y = -y^2$.......(F).

Remark : $y = 0$ is a solution of (F).

If $y \neq 0$,we divide the equation (F) by y^2, we obtain

$(1-x^3)y'y^{-2} + 3x^2y^{-1} = -1$........$(F')$.

We set $Z = y^{-1}$, then $Z' = -y'y^{-2}$.

We substitute into the equation (F'), we find

$-(1-x^3)Z' + 3x^2Z = -1$...(F_ℓ) : it is a linear differential equation of order 1.

We solve this equation using the method of variation of the constant (see exercise 3), and we obtain

$$Z = \frac{K(x)}{1-x^3} = \frac{x+C}{1-x^3}, \qquad C \in \mathbb{R},$$

therefore, $y = \dfrac{1}{Z} = \dfrac{1-x^3}{x+C}$.

2.1.5 Riccati differential equations

Definition 66 *Riccati differential equations are written in the following form :*

$$y' + a(x)y = b(x)y^2 + c(x) \qquad (E).$$

where a, b and c are continuous functions on $I \subset \mathbb{R}$.

Solving method :

Let y_0 be a particular solution of (E).

By using the change of variable $u = y - y_0$, we transform the equation (E) into the form of a Bernoulli equation with $k = 2$:

$$u' + A(x)u = b(x)u^2.$$

Exercise 67 1) *Let the differential equation :*

$$2y' \cos x - 2y \sin x = y^2 ....(1).$$

a) *Specify the type of this equation.*

b) *Find the general solution of* (1).

2) *Let the differential equation (* ***Riccati equation****) defined by:*

$$2y' \cos x = y^2 + 2\cos^2 x - \sin^2 x ....(2)$$

c) *Check that.* $y_0 = \sin x$ *is a particular solution of* (2).

d) *By using the change of variable* $u = y - y_0$, *rewrite equation* (2)*in the form of* (1).

Then, deduce the general solution of (2).

Solution :

1) Let the differential equation :

$$2y'\cos x - 2y\sin x = y^2 ....(1)$$

a) It is a Bernoulli differential equation with. $k = 2$.

b) The general solution of (1) :

Remark : $y = 0$ is a solution of (1).

If $y \neq 0$, we divide the equation (1) by y^2, we obtain

$2y'y^{-2}\cos x - 2y^{-1}\sin x = 1........(1')$.

We set $Z = y^{-1}$, then $Z' = -y'y^{-2}$.

We substitute into the equation $(1')$, we find

$-2Z'\cos x - 2Z\sin x = 1...(E_\ell)$: it is a linear differential equation of order 1.

We solve this equation using the method of variation of the constant (see exercise 3), and we obtain

$Z = K(x)\cos x = (-\frac{1}{2}\tan x + C)\cos x = -\frac{1}{2}\sin x + C\cos x, \qquad C \in \mathbb{R},$

therefore, $y = \dfrac{1}{Z} = \dfrac{2}{-\sin x + C\cos x}$.

2) Let the differential equation (Riccati equation) defined by :

$$2y'\cos x = y^2 + 2\cos^2 x - \sin^2 x ....(2).$$

c) We check that $y_0 = \sin x$ is a particular solution of (2).

$y_0 = \sin x \Longrightarrow y_0' = \cos x,$

We substitute into the equationn (2) :

$2y_0'\cos x = {y_0}^2 + 2\cos^2 x - \sin^2 x \Longleftrightarrow 2\cos^2 x = 2\cos^2 x,$

then, $y_0 = \sin x$ is a particular solution of (2).

d) We set $u = y - y_0$ to rewrite equation (2) in the form of (1).

$u = y - y_0 \Longrightarrow y = u + y_0 = u + \sin x \Longrightarrow y' = u' + \cos x.$

We substitute into the equation (2) :

$2(u' + \cos x)\cos x = (u + \sin x)^2 + 2\cos^2 x - \sin^2 x,$

then, $2u'\cos x - 2u\sin x = u^2$: it is the equation (1).

Therefore, the general solution of this equation is given, according to question b), by

$u = \dfrac{2}{-\sin x + C\cos x}, \qquad C \in \mathbb{R},$

thus, $y = u + \sin x = \dfrac{2}{-\sin x + C\cos x} + \sin x$ is the general solution of (2).

2.2 Linear differential equations of second order

2.2.1 Linear differential equations of second order with constant coefficients

Definition 68 *Linear differential equations of second order with constant coefficients are written in the following form :*

$$y'' + ay' + by = f(x).......(E),$$

where $a, b \in \mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ is a continuous function.

Solving method :

The general solution of (E) is written in the form of : $y = y_p + y_0$,

where y_p is a particular solution of (E) and y_0 is the general solution of the equation without the right-hand side (E_0).

$y'' + ay' + by = 0....(E_0)$ is the equation without the right-hand side.

Let $r^2 + ar + b = 0.....(E_c)$ be the characteristic equation associated with (E_0).

Remark 69 $y = 0$ *is a solution of the differential equation* (E_0).

Solution of the equation without the right-hand side :

$y'' + ay' + by = 0....(E_0)$

Remark 70 *Let y_1 and y_2 be two solutions of the quation (E_0) and let C_1 and $C_2 \in \mathbb{R}$, then $y = C_1 y_1 + C_2 y_2$ is also a solution to the equation (E_0).*

We are looking for the solutions of the equation (E_0) in the form : $y = e^{rx}$, $r \in \mathbb{R}$.

By substituting into the equation (E_0), we obtain

$e^{rx}(r^2 + ar + b) = 0$,

hence, $r^2 + ar + b = 0.....(E_c)$.

- If $\Delta > 0$, we have two real roots r_1,$r_2 \in \mathbb{R}$, then we obtain two solutions : $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$.

In this case, the solution of (E_0) is written in the form : $y_0 = C_1 y_1 + C_2 y_2$,

$y_0 = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad C_1, C_2 \in \mathbb{R}$.

- If $\Delta = 0$, we have a double root r, then we obtain two solutions :
$y_1 = e^{rx}$ and $y_2 = xe^{rx}$.

In this case, the solution of (E_0) is written in the form :

$y_0 = C_1 e^{rx} + C_2 x e^{rx}, \quad C_1, C_2 \in \mathbb{R}$.

- If $\Delta < 0$, we have two complex and conjugate roots $r = \alpha \pm i\beta$, in this case, the solution of (E_0) is written in the form :

$y_0 = e^{\alpha x} \left(C_1 \cos(\beta x) + C_2 \sin(\beta x) \right), \quad C_1, C_2 \in \mathbb{R}$.

Solution of the equation with the right-hand side :

Let the equation $y'' + ay' + by = f(x)....(E)$.

The general solution of (E) is written in the form : $y = y_p + y_0$,

where y_p is a particular solution of (E) and y_0 is the general solution of the equation without the right-hand side $(E_0) : y'' + ay' + by = 0$.

Method 1 :

Let $y_0 = C_1 y_1 + C_2 y_2$ be the general solution of (E_0).

We are looking for the general solution of (E) using the method of variation of constants. This method involves replacing the constants C_1 and C_2 by the functions $C_1(x)$ and $C_2(x)$ in y_0.

We set $y = C_1(x) y_1 + C_2(x) y_2$.

We derive : $y' = C_1'(x) y_1 + C_1(x) y_1' + C_2'(x) y_2 + C_2(x) y_2'$.

We choose $C_1(x)$ and $C_2(x)$ such that : $\mathbf{C_1'(x) y_1 + C_2'(x) y_2 = 0....(1)}$,

hence, $y' = C_1(x) y_1' + C_2(x) y_2'$.

We derive : $y'' = C_1'(x) y_1' + C_1(x) y_1'' + C_2'(x) y_2' + C_2(x) y_2''$.

Then, we substitute into $(E) : y'' + ay' + by = f(x)$,

$C_1'(x) y_1' + C_1(x) y_1'' + C_2'(x) y_2' + C_2(x) y_2'' + a(C_1(x) y_1' + C_2(x) y_2') + b(C_1(x) y_1 + C_2(x) y_2) = f(x)$,

we obtain, $C_1(x)(y_1'' + ay_1' + by_1) + C_2(x)(y_2'' + ay_2' + by_2) + C_1'(x) y_1' + C_2'(x) y_2' = f(x)$.

Since $y_1'' + ay_1' + by_1 = 0$ and $y_2'' + ay_2' + by_2 = 0$, we get,

$\mathbf{C_1'(x) y_1' + C_2'(x) y_2' = f(x)....(2)}$.

From the equations (1) and (2), we obtain $C_1'(x)$ and $C_2'(x)$, then we integrate to find $C_1(x)$ and $C_2(x)$.

Therefore, to find $C_1(x)$ and $C_2(x)$, we solve the following system :

$$\begin{cases} C_1'(x) y_1 + C_2'(x) y_2 & = & 0 \\ C_1'(x) y_1' + C_2'(x) y_2' & = & f(x) \end{cases}$$

Method 2 : Special cases

1) We assume that the right-hand side of the equation (E) is written in the form (E) :

$$f(x) = P_n(x)e^{\lambda x},$$

where $P_n(x)$ is a polynomial of degree n and $\lambda \in \mathbb{R}$.

- If λ is not a root of the characteristic equation (E_c), the particular solution of the equation (E) is written in the form

$$y_p = Q_n(x)e^{\lambda x},$$

where $Q_n(x)$ is a polynomial of degree n to be determined.

- If λ is a simple root of the characteristic equation (E_c), the particular solution of the equation (E) is written in the form

$$y_p = xQ_n(x)e^{\lambda x}.$$

- If λ is a double root of the characteristic equation (E_c), the particular solution of the equation (E) is written in the form

$$y_p = x^2Q_n(x)e^{\lambda x}.$$

2) We assume that the right-hand side of the equation (E) is written in the form

$$f(x) = e^{\lambda x}\left(P_n(x)\cos(\omega x) + Q_m(x)\sin(\omega x)\right),$$

where $P_n(x)$ is a polynomial of degree n , $Q_m(x)$ is a polynomial of degree m and $\lambda, \omega \in \mathbb{R}$.

- If $\lambda+\omega i$ is not a root of the characteristic equation (E_c), then the particular solution of the equation (E) is written in the form

$$y_p = e^{\lambda x}\left[U_N(x)\cos(\omega x) + V_N(x)\sin(\omega x)\right]$$

where $N = \max(n, m)$, U_N and V_N are polynomials of degree N.

- If $\lambda + \omega i$ is a root of the characteristic equation (E_c), then the particular solution of the equation (E) is written in the form

$$y_p = xe^{\lambda x}\left[U_N(x)\cos(\omega x) + V_N(x)\sin(\omega x)\right].$$

2.2.2 Principle of superposition

If the differential equation is written in the form

$$y'' + ay' + by = f_1(x) + f_2(x),$$

then the solution of this equation is written in the following form :

$$y = y_0 + y_1 + y_2,$$

where,
y_0 is the general solution of the equation without the right-hand side :

$$y'' + ay' + by = 0, \qquad (E_0)$$

y_1 is the particular solution of the equation :

$$y'' + ay' + by = f_1(x), \qquad (E_1)$$

y_2 is the particular solution of the equation :

$$y'' + ay' + by = f_2(x). \qquad (E_2)$$

Exercise 71 *Let the following differential equation :*

$$y'' + 3y' + 2y = x + e^{-x}. \qquad (E)$$

1) *Find the general solution y_0 of the following equation without the right-hand side :*

$$y'' + 3y' + 2y = 0. \qquad (E_0)$$

2) *Find the particular solution y_1 of the equation :*

$$y'' + 3y' + 2y = x. \qquad (E_1)$$

3) *Find the particular solution y_2 of the equation:*

$$y'' + 3y' + 2y = e^{-x}. \qquad (E_2)$$

4) *Deduce the general solution of the equation (E).*

Solution :
Let the following differential equation :

$$y'' + 3y' + 2y = x + e^{-x}. \qquad (E)$$

1) We are looking for the general solution y_0 of the following equation without the right-hand side :

$$y'' + 3y' + 2y = 0. \qquad (E_0)$$

The characteristic equation : $r^2 + 3r + 2 = 0 \qquad (E_c)$,

$\Delta = 1 \Longrightarrow r_1 = -1$ and $r_2 = -2$,

then, $y_0 = C_1 e^{-x} + C_2 e^{-2x}, \quad C_1, C_2 \in \mathbb{R}$.

2) We are looking for the particular solution y_1 of the equation :

$$y'' + 3y' + 2y = x, \quad (E_1)$$

where $f_1(x) = x = P_1(x)e^{\lambda x}$,

with $\lambda = 0$ and $P_1(x) = x$ is a polynomial of degree 1.

$\lambda = 0$ is not a root of the characteristic equation (E_c), therefore, the particular solution of the equation (E_1) is written in the form

$$y_1 = Q_1(x)e^{\lambda x} = ax + b,$$

then, $y_1' = a$ and $y_1'' = 0$.

We substitute into the equation (E_1), we find

$3a + 2ax + 2b = x \Longrightarrow 2a = 1$ and $3a + 2b = 0 \Longrightarrow a = \frac{1}{2}$ and $b = -\frac{3}{4}$

therefore, $y_1 = \frac{1}{2}x - \frac{3}{4}$.

3) We are looking for the particular solution y_2 of the equation

$$y'' + 3y' + 2y = e^{-x}. \quad (E_2)$$

where $f_2(x) = e^{-x} = P_0(x)e^{\lambda x}$,

with $\lambda = -1$ and $P_0(x) = 1$ is a polynomial of degree 1.

$\lambda = -1$ is a simple root of the characteristic equation (E_c), therefore, the particular solution of the equation (E_2) is written in the form

$$y_2 = Q_0(x)e^{\lambda x} = xAe^{-x},$$

then, $y_2' = -xAe^{-x} + Ae^{-x} = (-Ax + A)e^{-x}$,

and $y_2'' = -(-Ax + A)e^{-x} + -Ae^{-x} = (Ax - 2A)e^{-x}$.

We substitute into the equation (E_2), we find
$(Ax - 2A)e^{-x} + 3(-Ax + A)e^{-x} + 2xAe^{-x} = e^{-x} \Longleftrightarrow A = 1$,

therefore, $y_2 = xe^{-x}$.

4) We deduce the general solution of the equation (E).

According to the principle of superposition, the general solution of the equation (E) is written in the form

$$y = y_0 + y_1 + y_2 = C_1 e^{-x} + C_2 e^{-2x} + \frac{1}{2}x - \frac{3}{4} + xe^{-x}, \quad C_1, C_2 \in \mathbb{R}.$$

2.3 Exercises

Exercise 72 *Solve the following differential equations, specifying the type :*

1) $xy' - 2y = x^4(1 + \tan^2 x)$.

Solution :

$xy' - 2y = x^4(1 + tg^2 x) \qquad (E)$:

It is a first-order linear differential equation.

First, we look for the solution of the equation without the right-hand side :

$(E_0) : xy' - 2y = 0$,

$y = 0$ is a solution of (E_0),

if $y \neq 0$, $\displaystyle\int \frac{dy}{y} = 2\int \frac{dx}{x} \Longrightarrow \ln|y| = 2\ln|x| + C, \quad C \in \mathbb{R}$,

then, $\ln|y| = \ln x^2 + C \Longrightarrow |y| = e^C x^2 \Longrightarrow y =^+_- e^C x^2 = Kx^2, \; K \in \mathbb{R}^*$.

Since $y = 0$ is a solution of (E_0), then, $y = Kx^2, \; K \in \mathbb{R}$.

The particular solution is not obvious, so we apply the method of variation of constants :

we set $y_G = K(x)x^2$ (It is the general solution of (E)), then

$y'_G = K'(x)x^2 + 2xK(x)$.

We replace in the equation $(E) : xy' - 2y = x^4(1 + \tan^2 x)$,

$K'(x)x^3 + 2x^2K(x) - 2K(x)x^2 = x^4(1 + \tan^2 x)$,

then, $K'(x) = x(1 + \tan^2 x) \Longrightarrow K(x) = \displaystyle\int x(1 + \tan^2 x)dx$.

We apply an integration by parts, we set

$$\begin{cases} U &= x \\ V' &= 1 + \tan^2 x \end{cases} \Longrightarrow \begin{cases} U' &= 1, \\ V &= \tan x, \end{cases}$$

$K(x) = x\tan x - \displaystyle\int \tan x dx = x\tan x + \ln|\cos x| + C, \quad C \in \mathbb{R}$.

therefore, $y_G = K(x)x^2 = x^3 \tan x + x^2 \ln|\cos x| + Cx^2, \quad C \in \mathbb{R}$.

Exercise 73 *Solve the following differential equations, specifying the type :*

$(x^3+1)y' - 3x^2y + xy^3 = 0.$

Solution :
$(x^3+1)y' - 3x^2y + xy^3 = 0 \quad (E):$

It's a first-order Bernoulli differential equation.

$y = 0$ is a solution de (E).

If $y \neq 0$, we divide by y^3, We find

$(x^3+1)y'y^{-3} - 3x^2y^{-2} + x = 0 \quad (E)'.$

We set $Z = y^{-2} \Longrightarrow Z' = -2y'y^{-3}$.

We replace in the equation $(E)'$ and we obtain

$-(x^3+1)\dfrac{Z'}{2} - 3x^2Z = -x \qquad (E\ell):$ It's a linear differential equation.

We first seek the solution of the equation without the right-hand side :

$(E\ell_0) : -(x^3+1)\dfrac{Z'}{2} - 3x^2Z = 0$, we obtain

$Z_0 = \dfrac{K}{(x^3+1)^2}, \quad K \in \mathbb{R}.$

The particular solution being not obvious, we apply the method of variation of the constant (MVC):

we set $Z_G = \dfrac{K(x)}{(x^3+1)^2}$.

We derive and substitute into $(E\ell)$, we get

$K(x) = \dfrac{2}{5}x^5 + x^2 + C, \quad C \in \mathbb{R},$

then, $Z_G = \dfrac{K(x)}{(x^3+1)^2} = \dfrac{\frac{2}{5}x^5 + x^2 + C}{(x^3+1)^2} = \dfrac{2x^5+5x^2+C}{5(x^3+1)^2},$

$y^2 = \dfrac{1}{Z} \Longrightarrow y = \pm\sqrt{\dfrac{5(x^3+1)^2}{2x^5+5x^2+C}}, \quad C \in \mathbb{R}.$

Exercise 74 *I) Let the differential equation be defined by*

$$(1-x^3)y' + 3x^2y = -y^2. \qquad (1)$$

1) *Give the type of this equation.*

2) *Find the general solution of* (1).

II) Let the differential equation be defined by :

$$(1-x^3)y' + x^2y + y^2 - 2x = 0. \qquad (2)$$

1) *Give the type of this equation.*

2) *Verify that* $y_0 = x^2$ *is a particular solution of* (2).

3) *By using the change of variable* $u = y - y_0$ *transform the equation* (2) *into the form* (1).

4) *Deduce the general solution of* (2).

Solution :

I) 1) $(1-x^3)y' + 3x^2y = -y^2 \quad (1),$

It's a first-order Bernoulli differential equation.

2) We are looking for the general solution of (1) :

if $y = 0$: it is a solution of (1),

if $y \neq 0$, we divide by y^2, we find

$(1-x^3)y'y^{-2} + 3x^2y^{-1} = -1. \qquad (1)'.$

We set $Z = y^{-1} \Longrightarrow Z' = -y'y^{-2}$.

By substituting into $(1)'$, we obtain

$-(1-x^3)Z' + 3x^2Z = -1 : (E\ell)$ (Linear differential equation),

$(E\ell_0) : -(1-x^3)Z' + 3x^2Z = 0$ (Separable differential equation),

the solution of $(E\ell_0)$ is given by

$Z = \dfrac{K}{1-x^3}, \; K \in \mathbb{R}.$

The particular solution being not obvious, we apply the method of variation of constant (MVC) :

We set $Z_G = \dfrac{K(x)}{1-x^3},$

$Z_G' = \dfrac{K'(x)(1-x^3) - K(x)(-3x^2)}{(1-x^3)^2}.$

We substitute into $(E\ell)$ and we find $K'(x) = 1$,

then, $K(x) = x + C, \; C \in \mathbb{R},$

hence, $Z_G = \dfrac{x+C}{1-x^3}.$

Finally, $y = \frac{1}{Z_G} = \dfrac{1-x^3}{x+C}.$

$II)$ 1) $(1-x^3)y' + x^2y + y^2 - 2x = 0 \qquad (2):$

It's a Riccati differential equation.

2) $y_0 = x^2 \Longrightarrow y_0' = 2x$: Verify the equation (2).

3) We set $u = y - y_0 \Longrightarrow y = u + x^2 \Longrightarrow y' = u' + 2x$.

By substituting into the equation (2), we obtain

$(1-x^3)u' + 3x^2u = -u^2$: it is the equation (1), then its solution is

$u = \dfrac{1-x^3}{x+C}$,

therefore, $y = u + x^2 = \dfrac{1-x^3}{x+C} + x^2 = \dfrac{1+Cx^2}{x+C}, \quad C \in \mathbb{R}$.

Exercise 75 *Solve the following differential equations of the second order :*

1) $y'' - 3y' + 2y = 0$.

2) $y'' + 2y' + 5y = 0$ *satisfying* $y(0) = 0$ *and* $y'(0) = 1$.

3) $y'' - 2y' + y = (x^2+1)e^x$.

4) $y'' - y' + y = 2x^2e^{-x}$.

5) $y'' - y = -6\cos x + 2\sin x$.

Solution :

1) $y'' - 3y' + 2y = 0$.

The characteristic equation : $r^2 - 3r + 2 = 0$,

$\Delta = 1 \Longrightarrow r_1 = 1$ et $r_2 = 2$,

then, $y_0 = C_1e^x + C_2e^{2x}, \quad C_1, C_2 \in \mathbb{R}$.

*****

2) $y'' + 2y' + 5y = 0$ satisfying $y(0) = 0$ and $y'(0) = 1$.

The characteristic equation : $r^2 + 2r + 5 = 0$,

$\Delta = -16 = 16i^2 \Longrightarrow r = -1 \pm 2i$,

then, $y_0 = e^{-x}\left(C_1\cos(2x) + C_2\sin(2x)\right), \quad C_1, C_2 \in \mathbb{R}$.

We are looking for the particular solution that satisfies $y(0) = 0$ and $y'(0) = 1$.

$y(0) = 0 \Longleftrightarrow C_1 = 0$,

then, $y_0 = C_2e^{-x}\sin(2x) \Longrightarrow y_0' = -C_2e^{-x}\sin(2x) + 2C_2e^{-x}\cos(2x)$,

$y'(0) = 1 \Longleftrightarrow C_2 = \frac{1}{2}$,

hence, $y_0 = \frac{1}{2}e^{-x}\sin(2x)$.

*****

3) $y'' - 2y' + y = (x^2+1)e^x$......$(E)$.

We begin by solving the equation without the right-hand side :

$y'' - 2y' + y = 0$......(E_0).

The characteristic equation : $r^2 - 2r + 1 = 0$,

$\Delta = 0 \Longrightarrow r = 1$: it is a double root,

then, $y_0 = C_1e^x + C_2xe^x, \quad C_1, C_2 \in \mathbb{R}$.

We now seek the general solution of (E) using the method of variation of constants. This method consists of replacing the constants C_1 et C_2 by the functions $C_1(x)$ and $C_2(x)$.

We set $y = C_1(x)e^x + C_2(x)xe^x = C_1(x)y_1 + C_2(x)y_2$,

where $y_1 = e^x$ and $y_2 = xe^x$.

To find $C_1(x)$ and $C_2(x)$, we solve the following system:

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 &= 0, \\ C_1'(x)y_1' + C_2'(x)y_2' &= f(x), \end{cases}$$

$$\Longleftrightarrow \begin{cases} C_1'(x)e^x + C_2'(x)xe^x &= 0, \\ C_1'(x)e^x + C_2'(x)(x+1)e^x &= (x^2+1)e^x, \end{cases}$$

$$\Longleftrightarrow \begin{cases} C_1'(x) + C_2'(x)x &= 0, \\ C_1'(x) + C_2'(x)(x+1) &= x^2+1, \end{cases}$$

$$\Longleftrightarrow \begin{cases} C_1'(x) &= -C_2'(x)x \\ C_2'(x) &= x^2+1 \end{cases} \Longleftrightarrow \begin{cases} C_1'(x) &= -x^3 - x, \\ C_2'(x) &= x^2+1, \end{cases}$$

hence

$$\begin{cases} C_1(x) &= -\frac{x^4}{4} - \frac{x^2}{2} + K_1, \\ C_2(x) &= \frac{x^3}{3} + x + K_2, \end{cases}$$

then, $y = C_1(x)e^x + C_2(x)xe^x = (-\frac{x^4}{4} - \frac{x^2}{2} + K_1)e^x + (\frac{x^3}{3} + x + K_2)xe^x$,

thus, $y = (\frac{x^4}{12} + \frac{x^2}{2} + K_1 + K_2x)e^x$ is the general solution of (E).

*****

4) $y'' - y' + y = 2x^2e^{-x}$.....$(F)$.

We begin by solving the equation without the right-hand side :

$y'' - y' + y = 0$......(F_0).

The characteristic equation : $r^2 - r + 1 = 0$...(F_c),

$\Delta = -3 = 3i^2 \Longrightarrow r = \dfrac{1 \pm i\sqrt{3}}{2}$,

then, $y_0 = e^{\frac{1}{2}x}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right), \quad C_1, C_2 \in \mathbb{R}$.

We now seek a particular solution of (F) using the second method : the right-hand side of the equation (F) is written in the form

$$f(x) = 2x^2e^{-x} = P_2(x)e^{\lambda x},$$

where $\lambda = -1$ and $P_2(x) = 2x^2$ a polynomial of degree 2.

$\lambda = -1$ is not a root of the characteristic equation (F_c), then the particular solution of the equation (F) is written in the form

$$y_p = Q_2(x)e^{\lambda x} = (ax^2 + bx + c)e^{-x}.$$

We calculate y_p' and y_p'' :

$y_p' = -(ax^2 + bx + c)e^{-x} + (2ax + b)e^{-x} = (-ax^2 + (2a - b)x + b - c)e^{-x}$,

$y_p'' = -(-ax^2 + (2a - b)x + b - c)e^{-x} + (-2ax + 2a - b)e^{-x}$

$= (ax^2 + (-4a + b)x + 2a - 2b + c)e^{-x}$.

We substitute y_p' and y_p'' in the equation (F), we obtain

$(ax^2 + (-4a + b)x + 2a - 2b + c)e^{-x} - (-ax^2 + (2a - b)x + b - c)e^{-x} + (ax^2 + bx + c)e^{-x} = 2x^2e^{-x}$

$\Longleftrightarrow 3ax^2 + (-6a + 3b)x + 2a - 3b + 3c = 2x^2$,

by identification, we find

$$\begin{cases} 3a & = & 2 \\ -6a + 3b & = & 0 \\ 2a - 3b + 3c & = & 0 \end{cases} \Longleftrightarrow \begin{cases} a & = & 2/3, \\ b & = & 4/3, \\ c & = & 8/9, \end{cases}$$

then, $y_p = \left(\dfrac{2}{3}x^2 + \dfrac{4}{3}x + \dfrac{8}{9}\right)e^{-x}$,

hence, the general solution of (F) is $y = y_p + y_0$,

$y = \left(\dfrac{2}{3}x^2 + \dfrac{4}{3}x + \dfrac{8}{9}\right)e^{-x} + e^{\frac{1}{2}x}\left(C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$,

$C_1, C_2 \in \mathbb{R}$.

*****

5) $y'' - y = -6\cos x + 2\sin x$.....$(G)$.

We begin by solving the equation without the right-hand side

$y'' - y = 0$......(G_0).

The characteristic equation : $r^2 - 1 = 0$...(G_c),

$r^2 = 1 \Longrightarrow r_1 = 1$ and $r_2 = -1$,

then, $y_0 = C_1 e^x + C_2 e^{-x}, \quad C_1, C_2 \in \mathbb{R}$.

Now, we seek a particular solution of (G) using the second method: the right-hand side of the equation (G) is written in the form

$f(x) = e^{\lambda x}[P_0(x)\cos(\omega x) + Q_0(x)\sin(\omega x)] = -6\cos x + 2\sin x$,

with $\lambda = 0$, $\omega = 1$, $P_0(x) = -6$ and $Q_0(x) = 2$ polynomials of degree 0.

$\lambda + \omega i = i$ is not a root of the characteristic equation (G_c), therefore, the particular solution of the equation (G) is written in the form

$$y_p = e^{\lambda x}[U_0(x)\cos(\omega x) + V_0(x)\sin(\omega x)] = A\cos(x) + B\sin(x).$$

We calculate y_p' and y_p'' :

$y_p' = -A\sin x + B\cos x$,

$y_p'' = -A\cos x - B\sin x$.

We substitute y_p' and y_p'' in the equation (G), we obtain

$-A\cos x - B\sin x - A\cos x - B\sin x = -6\cos x + 2\sin x$

$\Longrightarrow -2A\cos x - 2B\sin x = -6\cos x + 2\sin x$,

by identification, we find

$$\begin{cases} -2A &= -6 \\ -2B &= 2 \end{cases} \Longleftrightarrow \begin{cases} A &= 3, \\ B &= -1, \end{cases}$$

then, $y_p = 3\cos(x) - \sin(x)$,

hence, the general solution of (G) is

$$y = y_p + y_0 = 3\cos(x) - \sin(x) + C_1 e^x + C_2 e^{-x}, \quad C_1, C_2 \in \mathbb{R}.$$

Chapter 3

Usual formulas

3.1 Partial sum of an arithmetic sequence

$U_n = U_0 + nr, \;\; r \in \mathbb{R}^*$.

$S_n = U_0 + U_1 + U_2 + ... + U_n = (U_0 + U_n)\dfrac{n+1}{2}$.

3.2 Partial sum of a geometric sequence

$U_n = U_0 q^n, \;\; q \neq 1$,

$S_n = U_0 + U_1 + U_2 + ... + U_n = U_0\left(\dfrac{1-q^{n+1}}{1-q}\right)$.

If $q = 1, S_n = (n+1)U_0$.

$\lim\limits_{n \longrightarrow +\infty} q^n = 0 \Longleftrightarrow -1 < q < 1$.

3.3 Trigonometry Formulas

1) $\sin(a+b) = \sin a \cos b + \sin b \cos a$, so $\sin 2a = 2 \sin a \cos a$.

2) $\sin(a-b) = \sin a \cos b - \sin b \cos a$.

3) $\cos(a+b) = \cos a \cos b - \sin a \sin b$, so $\cos 2a = \cos^2 a - \sin^2 a$.

4) $\cos(a-b) = \cos a \cos b + \sin a \sin b$.

5) $\cos 2a = 2\cos^2 a - 1$, so $\cos^2 a = \dfrac{\cos 2a + 1}{2}$.

6) $\cos 2a = 1 - 2\sin^2 a$, so $\sin^2 a = \dfrac{1 - \cos 2a}{2}$.

7) $\sin p + \sin q = 2\sin\frac{p+q}{2}\cos\frac{p-q}{2}$.

8) $\sin p - \sin q = 2\sin\frac{p-q}{2}\cos\frac{p+q}{2}$.

9) $\cos p + \cos q = 2\cos\frac{p-q}{2}\cos\frac{p+q}{2}$.

10) $\cos p - \cos q = -2\sin\frac{p-q}{2}\sin\frac{p+q}{2}$.

11) $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a.\tan b}$.

12) $\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a.\tan b}$.

Relation between sine and cosine
$\sin^2 x + \cos^2 x = 1, \quad \forall x \in \mathbb{R}$.

3.4 Common values

Number	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π
sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		0

3.5 Properties of hyperbolic functions

Hyperbolic sine : $shx = \frac{e^x - e^{-x}}{2}, \ \forall x \in \mathbb{R}$.

Hyperbolic cosine : $chx = \frac{e^x + e^{-x}}{2}, \ \forall x \in \mathbb{R}$.

1) $chx + shx = e^x$.

2) $chx - shx = e^{-x}$.

3) $ch^2x - sh^2x = 1$.

4) $ch(x+y) = chx.chy + shx.shy$.

5) $ch(2x) = ch^2x + sh^2x = 1 + 2sh^2x = 2ch^2x - 1$.

6) $sh(x+y) = shx.chy + shy.chx$.

7) $sh(2x) = 2shx.chx$.

3.6 Derivatives of usual functions

The function	The derivative
$f(x) = x^n$	$f'(x) = nx^{n-1}, \forall x \in \mathbb{R}$
$f(x) = \ln x$	$f'(x) = \frac{1}{x}, \forall x > 0$
$f(x) = e^x$	$f'(x) = e^x, \forall x \in \mathbb{R}$
$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}, \forall x > 0$
$f(x) = \sin x$	$f'(x) = \cos x, \forall x \in \mathbb{R}$
$f(x) = \cos x$	$f'(x) = -\sin x, \forall x \in \mathbb{R}$
$f(x) = \tan x = \frac{\sin x}{\cos x}$	$f'(x) = \frac{1}{\cos^2 x} = 1 + \tan^2 x, x \neq \frac{\pi}{2} + k\pi$
$f(x) = shx = \frac{e^x - e^{-x}}{2}$	$f'(x) = chx = \frac{e^x + e^{-x}}{2}, \forall x \in \mathbb{R}$
$f(x) = chx$	$f'(x) = shx, \forall x \in \mathbb{R}$
$f(x) = thx = \frac{shx}{chx}$	$f'(x) = \frac{1}{ch^2 x} = 1 - th^2 x, \forall x \in \mathbb{R}$
$f(x) = \arcsin x, \forall x \in [-1, 1]$	$f'(x) = \frac{1}{\sqrt{1-x^2}}, \forall x \in]-1, 1[$
$f(x) = \arccos x, \forall x \in [-1, 1]$	$f'(x) = \frac{-1}{\sqrt{1-x^2}}, \forall x \in]-1, 1[$
$f(x) = \arctan x$	$f'(x) = \frac{1}{1+x^2}, \forall x \in \mathbb{R}$
$f(x) = \arg shx$	$f'(x) = \frac{1}{\sqrt{x^2+1}}, \forall x \in \mathbb{R}$
$f(x) = \arg chx, \forall x \geq 1$	$f'(x) = \frac{1}{\sqrt{x^2-1}}, \forall x > 1$
$f(x) = \arg thx, \forall x \in]-1, 1[$	$f'(x) = \frac{1}{1-x^2}, \forall x \in]-1, 1[$
$f(x) = (U(x))^n$	$f'(x) = nU'(x)U^{n-1}(x)$
$f(x) = \ln(U(x))$	$f'(x) = \frac{U'(x)}{U(x)}$
$f(x) = e^{ax}$	$f'(x) = ae^{ax}, \forall x \in \mathbb{R}$

3.7 Antiderivatives of usual functions

The function	The antiderivative
$f(x) = x^n$	$\int f(x)dx = \frac{x^{n+1}}{n+1} + C, \forall x \in \mathbb{R}$
$f(x) = \ln x$	$\int f(x)dx = x \ln x - x + C, \forall x > 0$
$f(x) = e^x$	$\int f(x)dx = e^x + C, \forall x \in \mathbb{R}$
$f(x) = \sqrt{x}$	$\int f(x)dx = \frac{2}{3}x^{3/2} + C, \forall x > 0$
$f(x) = \sin x$	$\int f(x)dx = -\cos x + C, \forall x \in \mathbb{R}$
$f(x) = \cos x$	$\int f(x)dx = \sin x + C, \forall x \in \mathbb{R}$
$f(x) = \tan x = \frac{\sin x}{\cos x}$	$\int f(x)dx = -\ln\lvert\cos x\rvert + C-, x \neq \frac{\pi}{2} + k\pi$
$f(x) = shx = \frac{e^x - e^{-x}}{2}$	$\int f(x)dx = chx + C = \frac{e^x + e^{-x}}{2} + C, \forall x \in \mathbb{R}$
$f(x) = chx = \frac{e^x + e^{-x}}{2}$	$\int f(x)dx = shx + C, \forall x \in \mathbb{R}$
$f(x) = thx = \frac{shx}{chx}$	$\int f(x)dx = \ln(chx) + C, \forall x \in \mathbb{R}$
$f(x) = U^n(x)U'(x)$	$\int f(x)dx = \frac{U^{n+1}(x)}{n+1} + C$
$f(x) = \sin(ax), a \neq 0$	$\int f(x)dx = -\frac{\cos(ax)}{a} + C, \forall x \in \mathbb{R}$
$f(x) = e^{ax}, a \neq 0$	$\int f(x)dx = \frac{e^{ax}}{a} + C, \forall x \in \mathbb{R}$

3.8 Lexicon

A

- Absolute value : valeur absolue.
- Absolute convergence : convergence absolue.
- Almost : presque.
- Analysis : analyse.
- Antisymetric : antisymétrique.
- Apex : sommet.
- Argument : argument.
- Arithmetic : arithmétique.
- Array : tableau.
- Assume : supposer.
- Assumption : supposition.
- Axiom : axiome.
- Axis : axe.

B

- Basis : base.
- Bijective : bijective.
- Bounded : borné.
- Bracket : parenthèse.
- By induction : par récurrence.

C

- Calculus : calcul.
- Cartesian coordinate system.: Repère cartésien.
- Cauchy sequence : suite de Cauchy.
- Center : centre
- Characteristic : caractéristique.
- Characteristic polynomial : polynôme caractéristique.
- Circle : cercle.
- Closed : fermé.
- Coefficient : coefficient.
- Combination : combinaison.
- Common factor : facteur commun.
- Commutative : commutatif.
- Complete : complet.
- Complex number : nombre complexe.
- Computation : calcul.
- Consequently : par conséquent.
- Constant : constante.
- Continuity : continuité.
- Continuous (function) : continue (fonction).
- Contraction : contraction.
- Convergence : convergence.
- Converge to a limit : converger vers une limite.
- Converse of a theorem : réciproque d'un théorème.

- Conversely : réciproquement.
- Coordinate : coordonnée.
- Cosine : cosinus.
- Countable : dénombrable.
- Counterexample : contre-exemple.
- Coverage of a set : recouvrement d'un ensemble.
- Cube root : racine cubique.
- Curve : courbe.

D

- Decomposition : décomposition.
- Decreasing function : fonction décroissante.
- Defined : défini.
- Degree : degré.
- Delete (to) : supprimer.
- Denote : noter.
- Density : densité.
- Derivative : dérivée.
- Direct sum : somme directe.
- Divide : diviser.
- Dot : point.

E

- Eigenvalue : valeur propre.
- Eigenvector : vecteur propre.
- Element : élément.
- Endpoint : Extrémité.
- Entire function : fonction entière.
- Equality : égalité.
- Equation : équation.
- Equilateral triangle : triangle equilatéral.
- Equivalence relation : relation d'équivalence.
- Equivalent : équivalent
- Euclidean : euclidien.
- Even : pair.
- Everywhere : partout.
- Exact : exact.
- Example : exemple.
- Exponential : exponentiel.

F

- Factorial : factoriel.
- Factorise : factoriser.
- Field : corps.
- Finite : fini.
- Finite dimensional real vector space : espace vectoriel réel de dimension finie
- Fixed : fixe.
- Fixed point : point fixe.

- Floor function : fonction partie entière.
- Formula : formule.
- Fractional line : trait de fraction.
- Free : libre.
- Function : fonction.
- Fundamental : fondamental.

G

- Graph : graphe.
- Greatest : plus grand (le).
- Greatest common divisor (gcd) : pgcd.
- Group : groupe.

H

- Higher derivative : dérivée d'ordre supérieur.
- Homogeneous : homogène.
- However : toutefois.
- Hyperbola : hyperbole.
- Hypotenuse : hypoténuse.
- Hypothesis : hypothèse.

I

- Identity : identité.
- Identity element : élément neutre.
- If and only if : si et seulement si.
- Increasing function : fonction croissante.
- Indeed : en effet.
- Independent : indépendant.
- Induction : récurrence.
- Inequality : inégalité.
- Infimum (greatest lower bound) : borne inférieure.
- Infinite : infini.
- Integer number : nombre entier.
- Integral : intégrale.
- Intermediate value theorem : théorème des valeurs intermédiaires.
- Interval : intervalle.
- inverse image : image réciproque.
- Invertible : inversible.
- Involve : impliquer.
- Irreducible : irréductible.
- Isocel triangle : triangle isocèle
- Isolated : isolé.
- Isomorphism : isomorphisme.

J

K

- Kernel : noyau.

L

- Law of composition : loi de composition.
- Least : plus petit.

- Least common multiple (lcm) : ppcm.
- Lemma : lemme.
- Length : longueur.
- Less than : plus petit que
- Let.....be : soit.
- Limit : limite
- Linear : linéaire.
- Linearly independent family : famille libre.
- Lower limit : limite inférieure.
- Lower bound : minorant.

M

- Major : majeur.
- Majorized : majoré
- Manifold : variété.
- Map : application.
- Maximal : maximal.
- Mean : moyenne.
- Meet of two sets : intersection de deux ensembles.
- Merely : seulement.
- Minimal : minimal.
- Minorized : minoré.
- Monic : unitaire.
- Monotonic function : fonction monotone.
- Multiplicity : multiplicité.
- Multiply : multiplier.

N

- Necessary condition : condition nécessaire.
- Negligible : négligeable.
- Neighborhood : voisinage.
- Neperian logarithm : logarithme népérien.
- Non-empty : non vide.
- Not all zero : non tous nuls.
- Null : nul.
- Number : nombre.
- Numerator : numérateur.

O

- Object : objet.
- Odd : impair.
- One-to-one map : application injective.
- Onto (a map) : surjective.
- Open : ouvert.
- Operator : opérateur.
- Order : ordre.
- Order or multiplicity of a root : ordre de multiplicité d'une racine.
- Order relation : relation d'ordre.
- Ordinate : ordonnée.

P
- Parameter : paramètre
- Partial fraction expansion : décomposition en éléments simples.
- Partial order : relation d'ordre.
- Partition : partition.
- Perfect : parfait.
- Period : période.
- Periodicity : périodicité.
- Permutation : permutation.
- Plane : plan.
- Point : point.
- Polynomial : polynôme.
- Power : puissance.
- Prime : premier.
- Prime number : nombre premier.
- Product : produit.
- Proof : preuve.
- Proper : propre.
- Property : propriété.
- Pythagorean triple : triplet pythagoricien.

Q

R
- Radius : rayon
- Raise to the power n : élever à la puissance n.
- Range : image.
- Rank : rang.
- Ratio : rapport.
- Rational function : fonction rationnelle.
- Real number : nombre réel.
- Rectangle : rectangle.
- Reduced : réduit.
- Regular : régulier
- Relatively prime integers : entiers premiers entre eux.
- Remark : remarque.
- representation : représentation.
- Right-hand side : membre de droite.
- Ring : anneau.
- Root : racine.
- Row : ligne.
- Rule : règle.
- Ruler : règle (instrument).

S
- Scalar : scalaire.
- Schwarz inequality : inégalité de Schwarz.
- Section : section.
- Segment : segment.

- Sequence : suite.
- Series : série.
- Set : ensemble.
- Several : plusieurs.
- Shape : forme.
- Sign : signe.
- Sine : sinus.
- Singular : singulier.
- Size : taille.
- Small : petit.
- Smooth : lisse.
- Space : espace.
- Square : élever au carré.
- Square : carré.
- Square root : racine carré.
- Star : Etoile.
- Strictly : strictement
- Sub : sous-
- Subgroup : sous-groupe.
- Subset : sous-ensemble (partie).
- Subspace : sous-espace.
- Subtract : soustraire.
- Subtraction : soustraction.
- Sufficient : suffisant.
- Sufficient condition : condition suffisante.
- Sum : somme.
- Summarize (to) : résumer.
- Support : support.
- Supremum (least upper bound) : borne supérieure.
- Surface : surface.
- Symmetric : symétrique.
- Symmetry : symétrie.
- System of linear equations : système d'équations linéaires.

T

- Tangent : tangente.
- Term : terme.
- Theorem : théorème.
- Theory : théorie.
- Totally ordered set : ensemble totalement ordonné.
- Trace : trace.
- Trajectory : trajectoire.
- Transform : transformation.
- Transitive : transitif.
- Translation : translation.
- Transpose : transposé.
- Trapezoid : trapèze.

- Triangle : triangle.
- Triangle inequality : inégalité triangulaire.
- Trivial : trivial.
- Type : type.

U

- Uncountable : indénombrable.
- Uniform continuity : continuité uniforme.
- Union : réunion.
- Universal : universel.
- Unknown : inconnue.
- Upper bound : majorant.

V

- Value : Valeur.
- Variable : variable.
- Vector : vecteur.
- Vector space : espace vectoriel.
- Volume : volume.

W

- Well-defined : bien défini.
- Width : largeur.
- Without loss of generality : sans perte de généralité.

X

Y

Z

- Zéro : zero.
- Zero of a polynomial : racine d'un polynôme.

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